The University of Melbourne School Mathematics Competition, 2025

SENIOR DIVISION
Solutions

1. Ten teams mix players as follows. The 1st team sends $\frac{1}{10}$ of their team members to the 2nd team. The 2nd team sends $\frac{1}{10}$ of their new total number of team members to the 3rd team. For example, if the 1st, 2nd and 3rd teams start with 20, 48 and 35 members then their members move as follows:

$$(20, 48, 35, ...) \mapsto (18, 50, 35, ...) \mapsto (18, 45, 40, ...)$$

The 3rd team sends $\frac{1}{10}$ of their new total number of team members to the 4th team, and so on, until the 10th team receives $\frac{1}{10}$ of the new total number of team members from the 9th team. Finally, the 10th team sends $\frac{1}{10}$ of their new total number of team members to the 1st team, and they find that all teams now have the same number of team members. If each team began with at least 1 and fewer than 100 members, then how many members do they end up with?

Solution: Let b_j be the number of team members in team j, for $j = 1, ..., 10$, before the teams are mixed. Let b_2' be the number of team members in the 2nd team after receiving members from the 1st team and before sending members to the 3rd team. Then to achieve equality

$$\frac{9}{10}b_2' = \frac{9}{10}\left(b_3 + \frac{1}{10}b_2'\right) \Rightarrow \frac{9}{10}b_2' = b_3$$

But this means that the number of members received by the 3rd team is the same as the number of members sent from the 3rd to 4th teams, and similarly each team sends exactly the number of members it received. Hence, to achieve equality there exists b such that $b_j = b$, for $j = 3, ..., 10$. Each team starts with b members, then has $\frac{10}{9}b$ members, then b members again.

The number of members of the 1st team satisfies

$$\frac{9}{10}b_1 + \frac{1}{10}\frac{10}{9}b = b \quad \Rightarrow \quad b_1 = \frac{80}{81}b$$

so the 1st team gives $\frac{1}{10}\frac{80}{81}b = \frac{8}{81}b$ members to the 2nd team, and

$$\frac{9}{10}\left(b_2 + \frac{8}{81}b\right) = b \quad \Rightarrow \quad b_2 = \frac{82}{81}b.$$

Choose $b = 81$ so that each of these numbers is an integer less than 100. The teams start with

$$80,\ 82,\ 81\ , ...,\ 81$$

members and end up with

$$81,\ 81,\ 81\ , ...,\ 81$$

members.

2. Find a set of distinct integers $\{a_1, ..., a_n\}$ which sum to 293 and have the property that for each a_j there exists a different element in the set $a_k \neq a_j$ such that the product $a_j \times a_k$ is the sum of the remaining integers in the set.

Solution:

$$ab = 293 - a - b \quad \Leftrightarrow \quad (a+1)(b+1) = 294$$

Since

$$294 = 2 \times 3 \times 7^2$$

it has factors

$$1, 2, 3, 6, 7, 14, 21, 42, 49, 98, 147, 294.$$

To find a and b, remove 1 and 294 and subtract 1 from each factor to get

$$1, 2, 5, 6, 13, 20, 41, 48, 97, 146.$$

The sum is 379 which is too large. Remove pairs that add to 86 (it is easy to argue that such at choice is unique):

$$86 = 5 + 48 + 13 + 20$$

which leaves

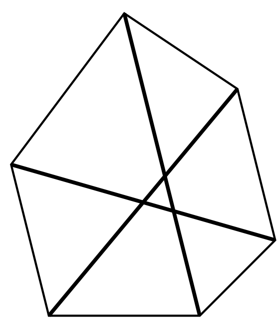
$$\{1, 2, 6, 41, 97, 146\}$$

$$1 + 2 + 6 + 41 + 97 + 146 = 293$$

$$1 \times 146 = 2 + 6 + 41 + 97, \quad 2 \times 97 = 1 + 6 + 41 + 146, \quad 6 \times 41 = 1 + 2 + 97 + 146.$$

There is also the trivial solution $\{0, 293\}$.

3. For any convex hexagon, define P to be its perimeter and D to be the sum of the lengths of its three main diagonals, as shown. Find the smallest real number α such that for all hexagons, $D < \alpha P$.



Solution: Any main diagonal has length less then $\frac{1}{2}P$ since traveling around the perimeter along three of the sides is longer than the direct route via the main diagonal. Thus twice the length of a main diagonal is bounded above by the perimeter. Apply this to all three main diagonals to get

$$D < \frac{3}{2}P.$$

To show that $\alpha = 3/2$ is the smallest such number, consider a thin hexagon so that the sum of the lengths of the main diagonals is approximately three times the length of the thin hexagon. Also, the perimeter is approximately two times the length of the thin hexagon. Then $D \approx \frac{3}{2}P$ and in fact by choosing a thin enough hexagon we can arrange that $D > (\frac{3}{2} - 0.1)P$, or that $D > (\frac{3}{2} - 0.01)P$ and so on. For any $\alpha < \frac{3}{2}$, we can choose a thin enough hexagon so that $D > \alpha P$, hence $\alpha = 3/2$ is the smallest number such that $D < \alpha P$.

4. Three people play the following game. They begin with one token each. A player is out when they have no tokens at the end of a turn and the winner is the final player remaining in. At each turn, a player is chosen with equal probability, and a token is taken from that player. Then, with equal probability this token is either removed from the game or given to any player remaining in, and the turn has finished. For example, at the first turn the probability that a token will be removed from the game is 1/4. The token may be taken from and returned to the same player. What is the expected number of tokens that the winner should end up with?

Solution:

Write E_{111} to be the expected number of tokens that the winner should end up with.

To begin, instead consider the same game with only two players and write E_{11} to be the expected number of tokens that the winner should end up with. Then

$$E_{11} = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot E_{11}$$

where the right hand side shows a token removed, respectively a token given to the other player, respectively a token given back to the player it was taken from. From this we solve to get $E_{11} = \frac{3}{2}$. Note that although it may take many turns to finish the game, the probability of this occurring is small, and in fact we see that with probability 1 the game will finish.

Similarly, if there are only two players, player one with one token and player two with two tokens, write E_{12} to be the expected number of tokens that the winner should end up with. Then

$$E_{12} = \frac{1}{2}\left(\frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot E_{12}\right) + \frac{1}{2}\left(\frac{1}{3} \cdot E_{11} + \frac{1}{3} \cdot E_{12} + \frac{1}{3} \cdot E_{12}\right)$$

where the first three terms show the outcomes of taking a token from player one, and removing it from the game, giving it to player two, giving it back to player one. The next three terms show the outcomes of taking a token from player two, and removing it from the game, giving it to player one, giving it back to player two. Hence

$$E_{12} = \frac{5}{6} + \frac{1}{2}E_{12} + \frac{1}{6}E_{11} = \frac{13}{12} + \frac{1}{2}E_{12} \quad \Rightarrow \quad E_{12} = \frac{13}{6}$$

Beginning with three players, we have

$$E_{111} = \frac{1}{4}E_{11} + \frac{1}{2}E_{12} + \frac{1}{4}E_{111}$$

which shows a token removed, respectively a token given to another player, respectively a token given back to the player it was taken from. Hence

$$E_{111} = \frac{3}{8} + \frac{13}{12} + \frac{1}{4}E_{111} \quad \Rightarrow \quad E_{111} = \frac{35}{18}.$$

5. Solve for x the following equation:

$$6 = \sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{64x + \cdots}}}}.$$

Solution:

$$1 + \sqrt{x} = \sqrt{(1 + \sqrt{x})^2} = \sqrt{x + 1 + \sqrt{4x}}.$$

Replace x with $4x$ to get $1 + \sqrt{4x} = \sqrt{4x + 1 + \sqrt{16x}}$ so that

$$1 + \sqrt{x} = \sqrt{x + \sqrt{4x + 1 + \sqrt{16x}}} = \sqrt{x + \sqrt{4x + \sqrt{16x + 1 + \sqrt{64x}}}}$$

and so on. Hence we claim* that

$$1 + \sqrt{x} = \sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{64x + \cdots}}}}.$$

Given this claim, then

$$1+\sqrt{x}=6 \quad \Rightarrow \quad x=25.$$

(*) To prove the claim, note that we have proven that the sequence

$$t_1=\sqrt{x}, \quad t_2=\sqrt{x+\sqrt{4x}}, \quad t_3=\sqrt{x+\sqrt{4x+\sqrt{16x}}}, \quad t_4=...$$

is increasing and bounded above by $1+\sqrt{x}$. We claim that the sequence reaches $1+\sqrt{x}$ in the limit. Define $\Delta_n(x)=1+\sqrt{x}-t_n$. Then

$$\Delta_n(x)\times(1+\sqrt{x}+t_n)=1+\sqrt{4x}-\sqrt{4x+\sqrt{16x}+...}=\Delta_{n-1}(4x)$$

and $1+\sqrt{x}+t_n>1+\sqrt{x}+\sqrt{x}=1+\sqrt{4x}$ so

$$\Delta_n(x)<\frac{\Delta_{n-1}(4x)}{1+\sqrt{4x}}<\frac{\Delta_{n-2}(16x)}{(1+\sqrt{4x})(1+\sqrt{16x})}<...<\frac{1}{(1+\sqrt{4x})(1+\sqrt{16x})...(1+\sqrt{4^{n-1}x})}$$

which shows that $\Delta_n(x)\to 0$ as $n\to\infty$ hence $t_n\to 1+\sqrt{x}$ as claimed.

6. Find all integers n that satisfy:

$$n \text{ is a factor of } 2^{2^n-1}-1.$$

Solution: Clearly $n=1$ gives a solution of

$$(*) \qquad n \text{ is a factor of } 2^{2^n-1}-1.$$

To find other solutions, consider odd $n>1$.

Let $p(n)$ be the smallest positive integer such that n divides $2^{p(n)}-1$. It exists and satisfies $1<p(n)<n$ because $2^1,2^2,...$ eventually repeats mod n and if n divides $2^{m+a}-2^a$ then n divides 2^m-1.

Now, if n divides 2^m-1 and $m\neq p(n)$ then n also divides $2^{m-p(n)}-1$ (equivalently $2^m=1=2^{p(n)}$ mod n implies $2^{m-p(n)}=1$ mod n). Thus $p(n)$ must divide m, since otherwise we can repeatedly apply $(m,p(n))\mapsto(m-p(n),p(n))$ (Euclid's algorithm) to find $0<m_1<p(n)$ such that n divides $2^{m_1}-1$. Conversely, if m is a multiple of $p(n)$ then n divides 2^m-1. So

$$n \text{ is a factor of } 2^{2^n-1}-1 \quad \Rightarrow \quad p(n) \text{ is a factor of } 2^n-1$$

but then $p(p(n))$ is a factor of n. Write $p_2(n):=p(p(n))$.

Note that by definition $p(n)$ is a factor of $2^{p_2(n)}-1$ hence

$$n \text{ is a factor of } 2^{2^{p_2(n)}-1}-1.$$

But since $p_2(n)$ is a factor of n we have

$$p_2(n) \text{ is a factor of } 2^{2^{p_2(n)}-1}-1.$$

Since $1<p(n)<n$, then $1<p_2(n)<n$ so given a solution $n>1$ of (*) we have produced a new strictly smaller solution to (*), still greater than 1. Repeating this will produce a strictly decreasing sequence of integers all greater than 1, which is a contradiction.

Hence $n=1$ is the only positive solution to (*). Also $n=0$ gives a solution.