The University of Melbourne School Mathematics Competition, 2025

INTERMEDIATE DIVISION
Solutions

1. Two ladies Mrs. A and Mrs. B live in separate towns. At the same time one day, each sets out on foot for the town of the other, walking at her own constant rate. They pass at noon. Mrs. A reaches her destination at 4 pm, while Mrs. B reaches her destination at 9 pm. At what time did they set out?

Solution:

Let's say that they set out t hours before noon. A walks for $(4+t)$ hours, B walks for $(9+t)$ hours. W.l.o.g. assume that the distance between their towns is $(4+t)(9+t)$ kilometres. Then A walks at $(9+t)$ km/hr, while B walks at $(4+t)$ km/hr. At noon they meet, so adding the distance the two ladies walked, one obtains the distance between the towns. So $(9+t)t+(4+t)t=(9+t)(4+t)$, which gives $t^2=36$. Clearly, we can discard the solution $t=-6$, as a non-existent time. So they started walking at 6 a.m.

2. Let p, q be consecutive prime numbers greater than 2. Show that $p+q$ is a product of three integers, each greater than 1. (The only factors of a prime number are 1 and itself, and an example of consecutive prime numbers is 23 and 29.)

Solution:

Since both p and q are odd, $p+q$ is even. Let $r=\frac{1}{2}(p+q)$. Since $p<q$, then

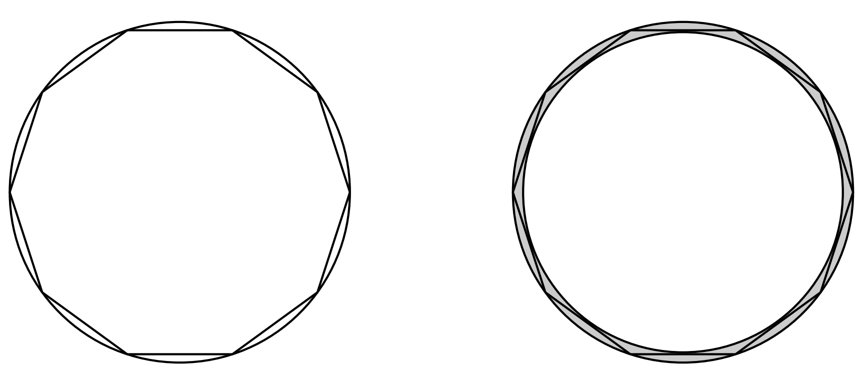
$$p<r<q.$$

But this means that r cannot be prime since there is no prime number between p and q. Thus $r=a\times b$ for some a and b both greater than 1. Then

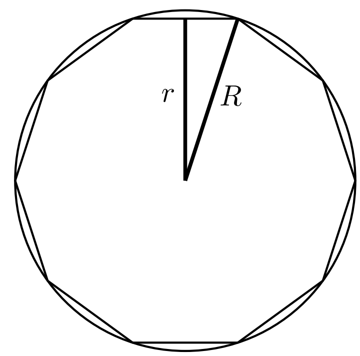
$$p+q=2\times a\times b$$

where each of 2, a and b are greater than 1.

3. Consider a regular decagon with sides of length 1cm inscribed inside a circle as in the left diagram. Inscribe a second circle inside the decagon. What is the area between the two circles? It is shaded in the right diagram.



Solution:



Denote the radius of the smaller circle by r and the radius of the larger circle by R and draw them as shown in the diagram. The area between the two circles is given by:

$$A = \pi(R^2 - r^2).$$

The two radii give two edges of a right angle triangle so that

$$r^2 + \left(\tfrac{1}{2}\right)^2 = R^2.$$

Hence the area in square centimetres is:

$$A = \pi(R^2 - r^2) = \pi\left(\tfrac{1}{2}\right)^2 = \frac{\pi}{4}.$$

4. Find all integer solutions of $x^3 + y^3 = 2025(x + y)$.

Solution:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = 2025(x + y).$$

Therefore

$$x + y = 0 \text{ or } (x^2 - xy + y^2) = 2025.$$

So we have the trivial solutions $(x, -x)$, for any integer x.

To solve $x^2 - xy + y^2 = 2025$, treat it as a quadratic equation in y, so the discriminant is

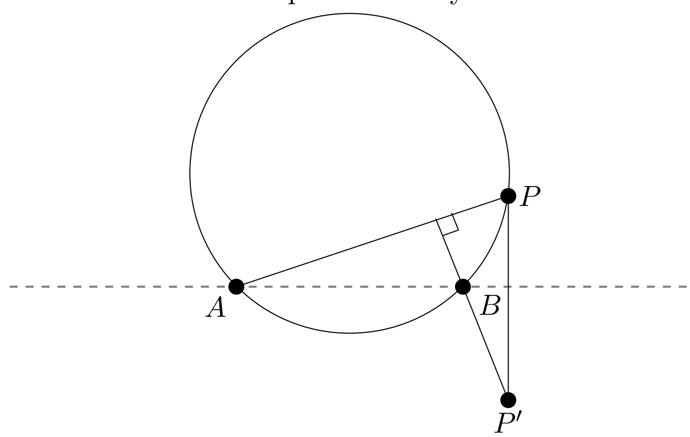
$$D = x^2 - 4(x^2 - 2025) = 3(2700 - x^2).$$

For a solution to exist, the discriminant must be positive, hence $|x| \leq \sqrt{2700} < 52$. Also, the square-root of the discriminant must be an integer, so $3(2700 - x) = k^2$. So k^2 must be divisible by 3, hence so must k. Write $k = 3m$. Then $x^2 = 3(900 - m^2)$ so $m^2 \leq 900$, so $m \leq 30$. Continuing, x^2 is divisible by 3, and so is x. Write $x = 3n$, giving $m = 900 - 3n^2$, or $|n| \leq \sqrt{300} \approx 17.32$. So $-17 < n < 17$, and we need $900 - 3n^2$ to be a perfect square. Trying all values from 0 to 17, we find only two, $n = 0$, and $n = 15$. The first value gives $(0, 45)$, $(0, -45)$. The second gives $(45, 0)$, $(-45, 0)$ and $(45, 45)$, $(-45, -45)$.

5. An elevator starts with 10 people on board and can stop at 10 floors. Each person picks a floor uniformly at random (independently). What's the expected number of floors that the elevator stops at?

Solution: For each floor $i \in [1, 10]$, let $X_i = 1$ if one person chooses that floor, and 0 otherwise. The total number of stops is $X = X_1 + X_2 + \cdots + X_{10}$. Each person chooses floor i with probability 0.1. So the probability that no-one chooses floor i is $P(X_i = 0) = (0.9)^{10}$. So the probability that at least on person chooses floor i is $P(X_i = 1) = 1 - (0.9)^{10}$. So the expected value $E[X_i] = 1 - 0.9^{10}$, and so $E(X) = \sum E[X_i] = 10(1 - 0.9^{10})$. (As $0.9^{10} \approx 0.3847$, $E(x) \approx 6.513$. That is, each floor has about a 65.1% chance of being selected.)

6. The diagram below shows points A and B on a horizontal line, and a circle containing these points. For any point P on the circle, let P' be the point directly below P such that the line through B and P' meets the line through A and P at right angles. Describe the path that P' takes as P moves around the circle.
Hint: It may help to show that B lies on the path taken by P'.



Solution:

Draw the point B' opposite to A on the circle. The angle $\angle APB'$ is a right angle since the large side length of triangle APB' is a diameter. For the same reason, B' lies directly above B since the angle $\angle ABB'$ must be a right angle. The vertical lines $B'B$ and PP' are parallel and the lines $B'P$ and FP' and BP' are parallel. Hence $BPP'B$ defines a parallelogram, hence the lengths of $B'B$ and PP' are equal. This holds for all P (the pictures differ slightly for P in the three arcs of the circle from B' to P to A to B' but the wording is the same.) In other words, the length of PP' is constant for all P. Hence the path that P' takes as P moves around the circle is a circle translated downwards by the distance the length of $B'B$.