The University of Melbourne School Mathematics Competition, 2025

SENIOR DIVISION

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The **six** questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

*Textbooks, electronic calculators and computers are **NOT** allowed. Otherwise normal examination conditions apply.*

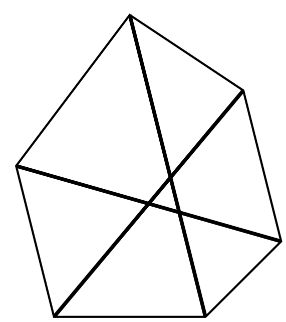
1. Ten teams mix players as follows. The 1st team sends $\frac{1}{10}$ of their team members to the 2nd team. The 2nd team sends $\frac{1}{10}$ of their new total number of team members to the 3rd team. For example, if the 1st, 2nd and 3rd teams start with 20, 48 and 35 members then their members move as follows:

$$(20, 48, 35, ...) \mapsto (18, 50, 35, ...) \mapsto (18, 45, 40, ...)$$

The 3rd team sends $\frac{1}{10}$ of their new total number of team members to the 4th team, and so on, until the 10th team receives $\frac{1}{10}$ of the new total number of team members from the 9th team. Finally, the 10th team sends $\frac{1}{10}$ of their new total number of team members to the 1st team, and they find that all teams now have the same number of team members. If each team began with at least 1 and fewer than 100 members, then how many members do they end up with?

2. Find a set of distinct integers $\{a_1, ..., a_n\}$ which sum to 293 and have the property that for each a_j there exists a different element in the set $a_k \neq a_j$ such that the product $a_j \times a_k$ is the sum of the remaining integers in the set.

3. For any convex hexagon, define P to be its perimeter and D to be the sum of the lengths of its three main diagonals, as shown. Find the smallest real number α such that for all hexagons, $D < \alpha P$.



4. Three people play the following game. They begin with one token each. A player is out when they have no tokens at the end of a turn and the winner is the final player remaining in. At each turn, a player is chosen with equal probability, and a token is taken from that player. Then, with equal probability this token is either removed from the game or given to any player remaining in, and the turn has finished. For example, at the first turn the probability that a token will be removed from the game is 1/4. The token may be taken from and returned to the same player. What is the expected number of tokens that the winner should end up with?

5. Solve for x the following equation:

$$6 = \sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{64x + \cdots}}}}.$$

6. Find all integers n that satisfy:

$$n \text{ is a factor of } 2^{2^n - 1} - 1.$$