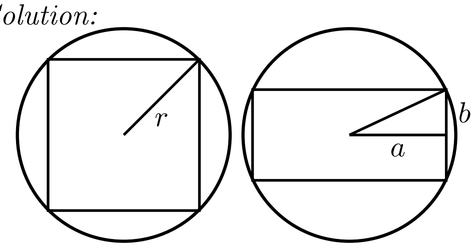
The University of Melbourne School Mathematics Competition, 2024

SENIOR DIVISION
Solutions

1. A rectangle inscribed in a circle has area equal to half the area of a square inscribed in the same circle. What is the ratio of the two side lengths of the rectangle?

Solution:



Let a and b be half the side lengths of the rectangle. By Pythagoras' theorem

$$a^2 + b^2 = r^2.$$

The area of the square is $2r^2$, hence the area of the rectangle is half of this:

$$r^2 = 4ab = a^2 + b^2 \Rightarrow x^2 - 4x + 1 = 0 \Rightarrow x = 2 \pm \sqrt{3}, \qquad \text{for } x = a/b.$$

Hence the ratio is $2 + \sqrt{3} : 1$ (or equivalently $1 : 2 - \sqrt{3}$).

2. In the city Neves, the year is written using base 7 in reverse order, although it still agrees with the year in the rest of the world. For example this year in Neves is written

$$N1265 = 1 \times 1 + 2 \times 7 + 6 \times 7^2 + 5 \times 7^3 = 2024$$

and $N0002 = 0 \times 1 + 0 \times 7 + 0 \times 7^2 + 2 \times 7^3 = 686$. When next will the year written in both Neves notation and decimal notation agree?

Solution 1: Assume that this coincidence occurs before the year 10000 in decimal notation, and if we do find a solution then our assumption is correct. If the year is written $Nabcd = abcd$ then we need

$$a \times 1 + b \times 7 + c \times 7^2 + d \times 7^3 = 1000a + 100b + 10c + d.$$

Note that the digits of a Neves year are each at most 6. The left hand side is bounded above by $6 \times (1 + 7 + 7^2 + 7^3) = 2400$ hence if a solution occurs, it occurs before 2400 and $a = 2$.

$$2024 - 6 \times (1 + 7 + 7^2) > 4 \times 7^3$$

so $d = 5$ or 6.

Since $Nabcd$ has a remainder of 2 when divided by 7, the same is true of $abcd$.

If $d = 5$ then

$$2 + b \times 7 + c \times 7^2 + 5 \times 7^3 \leq 2 + 6 \times 7 + 6 \times 7^2 + 5 \times 7^3 = 2053$$

so we only need to check 2025 which has a remainder of 2 when divided by 7, and ends in 5:

$$2025 = 2 \times 1 + 2 \times 7 + 6 \times 7^2 + 5 \times 7^3.$$

Hence $d = 6$. We only need to check

$$2046 + 70m$$

which has a remainder of 2 when divided by 7, and ends in 5 or 6. The list (with no digits greater than 6) is 2046, 2116, 2256, 2326.

Check

$$2046 = 2 \times 1 + 5 \times 7 + 6 \times 7^2 + 5 \times 7^3$$

$$2116 = 2 \times 1 + 1 \times 7 + 1 \times 7^2 + 6 \times 7^3$$

so we see that 2116 is the first year after 2024 when Neves notation and decimal notation agree.

In fact, by further adding $N0310 = 70$ we can also check

$$2256 = 2 \times 1 + 0 \times 7 + 4 \times 7^2 + 6 \times 7^3, \quad 2326 = 2 \times 1 + 3 \times 7 + 5 \times 7^2 + 6 \times 7^3$$

to see that 2116 is the *only* coincidence of the years before 10000.

Solution 2: As above, deduce that the first digit is 2. Since $Nabcd$ has a remainder of 2 when divided by 7, the same is true of $abcd$. We only need to check

$$2025 + 7k$$

which has a remainder of 2 when divided by 7. To shorten this list, we can also choose the second digit b and look at remainders when divided by 49, giving a list of two elements for each b.

Let's assume that $b = 0$ (for the earliest year) so that $abcd$ has a remainder of 2 when divided by 49. Then we only need to check

$$2053 > 2 + 5 \times 7^2 + 3 \times 7^3.$$

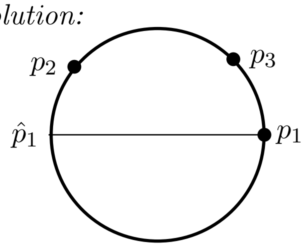
Instead assume that $b = 1$ so that $abcd$ has a remainder of 9 when divided by 49. Then we only need to check 2116 and 2165. We have

$$2116 = 2 \times 1 + 1 \times 7 + 1 \times 7^2 + 6 \times 7^3$$

which is the first year after 2024 when Neves notation and decimal notation agree.

3. Choose three points on a circle independently with uniform probability. What is the probability that the triangle with vertices given by the three points has an angle greater than ninety degrees?

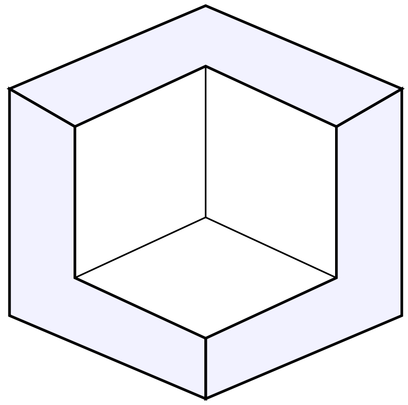
Solution:



Fix one point p_1, since by symmetry we can rotate any picture. Draw the diameter from p_1 to its opposite point $\hat{p}_1$, as in the picture. If p_2 and p_3 are in the same half then there is an angle greater than 90°, at the nearest of p_2 and p_3 to p_1, since it is exactly 90° if one of p_2 or p_3 coincides with $\hat{p}_1$. The probability that p_2 and p_3 are in the same half is $1/2 = 1/2 \times 1/2 + 1/2 \times 1/2$.

By symmetry the probablity that the angle at p_2 is greater than 90° is 1/4, hence the same is true of p_1. Thus the probability that the triangle with vertices given by the three points has an angle greater than ninety degrees is $1/2 + 1/4 = 3/4$.

4. From a cube of integer side length n, a smaller cube of integer side length m has been removed from one corner, as in the picture, so that as integers, the volume of the resulting shape is greater than its surface area by 3 units. Find all such m and n that give cubes with this property.



Solution:

$$m^3 - n^3 = 6m^2 + 3$$

Put $m = n + k$

$$(n+k)^3 - n^3 = 6(n+k)^2 + 3$$
$$\Rightarrow\ 3n^2k + 3nk^2 + k^3 - 6n^2 - 12nk - 6k^2 - 3 = 0$$
$$\Rightarrow\ (3k-6)n^2 + (3k^2 - 12k)n + k^3 - 6k^2 - 3 = 0$$

Hence 3 divides k and k is odd.

The discriminant is

$$\Delta = (3k^2 - 12k)^2 - 4(3k-6)(k^3 - 6k^2 - 3) = -3k^4 + 24k^3 + 36k - 72$$

and $\Delta < 0$ for $k \geq 9$ hence only $k = 3$ is possible.

$$8^3 - 5^3 = 6 \times 8^2 + 3$$

5. Over a ten week term, each student in a class of 25 students had to give two presentations to the class on a topic chosen by the teacher. Each day, one student was randomly chosen to give their presentation, so that after 5 weeks everyone had presented once. For the second half of the term, again the students were randomly chosen to give their presentations, although one student was missing due to illness and did not present. A student complained that they had less than a week between presentations, while others had more than four weeks between presentations. For each of the students who gave two presentations, the teacher counted the number of days between their two presentations and added these together to get the total number of days between presentations for the whole class. What is the maximum number of total days the teacher could have calculated?

Solution: If all students present twice then the total days is independent of the order. It is equal to

$$25 \times 35$$

(or 25×25 if we use 5 day weeks.)

To maximise, put the missing student in the last day of week 5 and the first day of week 6. Then the total is:

$$25 \times 35 - 2 = 873$$

(or $25 \times 25 = 625$.)

6. On the square integer grid consisting of all integer pairs (m, n) with $\max\{|m|, |n|\} \leq 2024$, there is a real valued function F satisfying the following conditions:

(1) if $\max\{|m|, |n|\} = 2024$, then

$$F(m, n) = \frac{mn}{1 + |mn|}$$

(2) if $\max\{|m|, |n|\} < 2024$, then

$$F(m, n) = \frac{1}{4}\Big(F(m+1, n) + F(m-1, n) + F(m, n+1) + F(m, n-1)\Big).$$

Prove that $F(m, n) > 0$ whenever $m > 0$ and $n > 0$.

Solution: Step 1: Let M be the maximum value of $F(m, n)$ on the square integer grid. Consider an interior point, i.e $\max\{|m|, |n|\} < 2024$ and suppose that $F(m, n) = M$ there. Then since $F(m, n)$ is the average of the values of its neighbours, then $F(m', n') = M$ for each of its neighbours. Repeat this reasoning to conclude that $F(m, n) = M$ for all points on the square integer grid. But this contradicts (1) above which shows that $F(m, n)$ takes on different values on the boundary of the grid, i.e when $\max\{|m|, |n|\} = 2024$. Hence the maximum value of $F(m, n)$ occurs on the boundary. Similarly, the minimum value of $F(m, n)$ also occurs on the boundary.

Step 2: Define $G(m, n) = F(m, n) + F(-m, n)$ on the same square grid. Then $G(m, n) = 0$ if $\max\{|m|, |n|\} = 2024$ and if $\max\{|m|, |n|\} < 2024$, then

$$G(m, n) = \frac{1}{4}\Big(G(m+1, n) + G(m-1, n) + G(m, n+1) + G(m, n-1)\Big).$$

But from Step 1, we conclude that the maximum and minimum values of $G(m, n)$ occur on the boundary, hence they are 0, hence $G(m, n) = 0$ for all points on the square grid. We conclude that for $F(m, n) + F(-m, n) = 0$ for any point on the square grid, so in particular $F(0, n) = 0$ for all $|n| \leq 2024$. Similarly, we can conclude that $F(m, 0) = 0$ for all $|m| \leq 2024$ (using the function $H(m, n) = F(m, n) + F(m, n)$). In other words $F(m, n)$ vanishes on the x and y axes.

Step 3: Apply Step 1 to the first quadrant, to conclude that the minimum of $F(m, n)$ cannot occur at an interior point of the first quadrant, instead the minimum occurs on the boundary (of the first quadrant). The smallest boundary value is 0, **hence $\mathbf{F(m, n) > 0}$ in the first quadrant**. The strict inequality is due to the argument in Step 1, which concludes that an interior value of 0 implies all values in the first quadrant are 0, which contradicts some of the boundary values.