The University of Melbourne School Mathematics Competition, 2024

INTERMEDIATE DIVISION
Solutions

1. Find the area of the shaded square in the figure below. (See question for figure.)

Solution. Denote the side length of the square by a. Form a right-angled triangle with hypotenuse given by a radius of the circle, and vertices the top left corner of the square, the centre of the left circle, and a third point to make a right angle triangle. The triangle has side lengths $r, r-a, r-a/2$. Apply Pythagoras' theorem to this square:

$$r^2 = (r-a)^2 + (r-a/2)^2$$
$$\Rightarrow \frac{5}{4}a^2 - 3ar + r^2 = 0$$
$$\Rightarrow a = \frac{2}{5}(3r \pm 2r) = \frac{2}{5}r = 2\text{cm}$$

Therefore, the area of the square is 4cm^2.

2. If we randomly pick 4 vertices from a cube, what is the probability that the chosen vertices lie in the same plane?

Solution. There are 6 faces and 8 vertices. There are 8 choose 4 ways of choosing four vertices in a plane, and there are 12 planes, 6 being faces of the cube and 6 being bisecting planes. So the answer is $12/(8\,\text{choose}\,4) = 6/35$.

If unfamiliar with such concepts, one can work it out from first principles. By symmetry, one can choose any vertex as the first vertex. There are 7 choices for the next vertex, 6 for the third and 5 for the fourth, giving a total of 210 choices. How many of these lie in a plane? Of the 7 remaining vertices, 1 is diagonally opposite in the cube, so cannot form part of a face, so let's first look at planes that are faces. 3 of the 6 remaining vertices are incident on the chosen vertex, and choosing any of these leaves four choices for the remaining two vertices to satisfy planarity, so a total of 12. The remaining 3 vertices are diagonally opposite the chosen vertex, but in the same plane, and there are then two choices of vertices that satisfy planarity, giving 6 possibilities. So of the 210 choices we have 18 lying in the same plane. Now consider planes that bisect the cube. By similar arguments, there are also 18 choices of vertices that give a plane that bisects the cube, so we have a total of 36 choices. So the required probability is 36/210=6/35.

3. In a small village, a survey of drinking preferences was carried out. It turns out that 90% of people drank coffee, 80% of people drank tea, 70% of people drank orange juice and 60% of people drank apple juice. Nobody drank all four beverages. What percentage of the village residents drank juice (either orange or apple)?

Solution. The percentages add to 300%. This means that the average person drinks three beverages. Since no one drinks more than three beverages, this means that everyone drinks exactly three beverages. That means that everyone excludes exactly one beverage. So none can exclude both juices, so 100% of people drink juice.

4. In the sum below, each letter represents a base 10 digit. No letter can represent more than one digit, and no digit is associated with more than one letter. There are no leading zeros. Find all solutions.

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  FAILURE
+FAILURE
+FAILURE
+FAILURE
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  SUCCESS
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Solution. Clearly, from the left-most column, $F = 1$ or $F = 2$.

Assume $\mathbf{F} = \mathbf{1}$. Then $S = 4$, 5, 6, 7. From the first column, S must be even, so $S = 4$ or 6.

But if $S = 4$, $E = 1$, all letters denote a unique digit, and we've assumed $F = 1$, so S must be 6.

This implies $E = 4$ or 9, but this implies $4R + 1 \bmod 10$ is 6 or $9R + 1 \bmod 10$ is 6, both of which are impossible. Hence $\mathbf{F} \neq \mathbf{1}$.

So $\mathbf{F} = \mathbf{2}$. Hence $S = 8$ or 9.

But $4E \bmod 10$ is S, so S must be even, so $\mathbf{S} = \mathbf{8}$ implying $E = 2$, or $E = 7$. But $F = 2$, so $\mathbf{E} = \mathbf{7}$. So $4R + 2 \bmod 10$ is 8, so $R = 4$ or $R = 9$.

If $R = 4$, then by similar arguments $U = 9$. If $U = 9$, then from the second last column, there must be an odd carry from the preceding column, so 1 or 3. It can't be 3 as we've already used 7, 8, 9, so it must be 1. But $U = 9$ implies $L = 3$ or $L = 5$ and these won't lead to a carry of 1. So $\mathbf{R} = \mathbf{9}$.

This implies $U = 1$ or $U = 6$. If $U = 6$, then $L = 3$ and $C = 4$ and there is no solution for I. So $\mathbf{U} = \mathbf{1}$. This allows for (L, C) pairs to be $(4, 6)$, $(5, 0)$, $(6, 4)$. The first two possibilities yield no solution for I so $\mathbf{L} = \mathbf{6}$ and $\mathbf{C} = \mathbf{4}$. This gives $4I + 2 \bmod 10$ is 4, so $\mathbf{I} = \mathbf{3}$. This implies $\mathbf{A} = \mathbf{0}$. So

$$(0, 1, 2, 3, 4, 6, 7, 8, 9) = (A, U, F, I, C, L, E, S, R).$$

This chain of argument leads to a unique solution.

5. The diameter of circle B is three times the diameter of circle A in the figure below. If circle A starts from the position shown, and rolls around the perimeter of circle B without slipping until it returns to its starting point, how many revolutions will it have made?

Solution. The "obvious" answer is three, and this would be true if the smaller coin were rolled along a linear path equal in length to the circumference of the larger circle. But there is an extra revolution involved by virtue of the circular path, so the answer is four revolutions. (This is a famous problem set in a US SAT multiple-choice test, in which all answers given were wrong!).

6. Prove that $14^n + 11$ is a composite number (that is, can never be a prime number), where n is a positive integer.

Solution 1. When $n = 1$, $14^n + 11 = 25$ which has a factor of 5, proving that it is composite. Although we insist that n is positive, neverthless consider $n = 0$, then $14^n + 11 = 12$ which has a factor of 3.

Claim: $14^n + 11$ always has a factor of 3 or 5, hence is composite.

Proof of claim: For any $n \geq 0$,

$$\begin{aligned} 14^2(14^n + 11) &= 14^{n+2} + 11 + (14^2 - 1) \times 11 = 14^{n+2} + 11 + (14 + 1) \times (14 - 1) \times 11 \\ &= 14^{n+2} + 11 + 15k \end{aligned}$$

so if $14^n + 11$ has a factor of 3 or 5 then $14^{n+2} + 11$ also has a factor of 3 or 5.

But we showed that $14^n + 11$ has a factor of 3 or 5 for $n = 0$ and $n = 1$, so this argument shows that $14^n + 11$ has a factor of 3 or 5 for all $n \geq 0$. (More precisely, it shows that $14^n + 11$ has a factor of 3 for n even and a factor of 5 for n odd.

Solution 2. When $n = 0$, $14^n + 11 = 12 = 3 \times 4$. When $n = 1$, $14^n + 11 = 25 = 5 \times 5$. When

$n = 2$, $14^n + 11 = 207 = 3 \times 69$. When $n = 4$, $14^n + 11 = 27455 = 5 \times 549$. So one might guess that for n even, 3 is a factor, and for n odd, 5 is a factor. To prove this, note that

$$14^n + 11 = (15 - 1)^n + 12 - 1 = 15k + (-1)^n + 12 - 1 = 3m + (-1)^n - 1 = 3m'$$

for n even, and is equal to $5m + 10 = 5(m + 2)$ for n odd.

Solution 3. Both 14 and 11 are equal to -1 modulo 3. Also, 14 and -11 are equal to -1 modulo 5.
So if n is even, then $14^n + 11 \equiv (-1)^n - 1 \equiv 0 \ (mod\ 3)$, so is divisible by 3, hence composite provided that $14^n + 11 > 3$, which it clearly is.
Similarly, if n is odd, than $14^n + 11 \equiv (-1)^n + 1 \equiv 0 \ (mod\ 5)$, so is divisible by 5, hence composite provided that $14^n + 11 > 5$, which it clearly is. Hence $14^n + 11$ is composite.

7. Find for what values of the non-negative integer n the number $10^n + 9$ is a perfect square.

Solution 1. From $10^n + 9 = x^2$, rewrite as

$$10^n = (x - 3)(x + 3).$$

Note that 5 cannot divide both $(x - 3)$ and $(x + 3)$. So one of these factors is at least 5^n and hence the other factor is at most 2^n i.e.

$$x + 3 \geq 5^n, \quad x - 3 \leq 2^n$$

But for $n > 1$

$$6 = x + 3 - (x - 3) \geq 5^n - 2^n > 6$$

which is a contradiction, hence $10^n + 9$ is never a perfect square for $n > 1$. When $n = 0$ and $n = 1$ we see that neither 10 nor 19 are not perfect squares, so there no values of n so that the number $10^n + 9$ is a perfect square

Solution 2. If n is even, then $10^n \equiv 1 \pmod{11}$, and if n is odd, then $10^n \equiv 10 \pmod{11}$. Therefore $10^n + 9 \equiv 10$ if n is even, and $10^n + 9 \equiv 8$ if n is odd. The squares, modulo 11 are

$$0 = 0^2, 1 = 1^2 = 10^2, 4 = 2^2 = 9^2, 9 = 3^2 = 8^2, 5 = 4^2 = 7^2, 3 = 5^2 = 6^2$$

which does not include 8 or 10. Thus we conclude that $10^n + 9$ is not a perfect square for any non-negative integer n.