Junior Solutions 2024

June 11, 2024

1. We show that Taylor was born in 1964. First notice that since each digit in Taylor's birth year is at most 9, she is at most $3 \times 4 \times 9 = 108$ years old. Hence Taylor was born after 1900. We split into two cases, based on the century Taylor was born.

   Case I: Taylor is born in the year $20ab$ for some digits a and b. Then Taylor is $24 - 10a - b$ years old, and also is $3(2+0+a+b)$ years old. Equating these and simplifying gives the equation

   $$13a + 4b = 6.$$

   Since a and b are non-negative integers, a must be zero as otherwise the left hand side is larger than 6. But then $b = 3/2$ which is not an integer, so there is no solution in this case.

   Case I: Taylor was born in the year $19ab$ for some digits a and b. Then Taylor is $124 - 10a - b$ years old, and also is $3(1+9+a+b)$ years old. Equating these and simplifying gives the equation

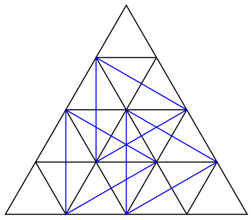
   $$13a + 4b = 94.$$

   Since $0 \leq b \leq 9$, we get $58 \leq 13a \leq 94$ and hence $4 < a < 8$. Since $4b$ and 94 are even and 13 is odd, it must be that a is even. Hence $a = 6$. Subsituting into the equation gives $b = 4$ which we check is indeed a solution.

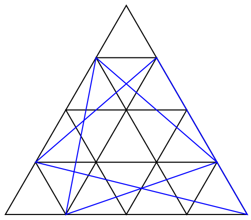
2. The answer is 35.

   Without loss of generality, say that the side length of the large triangle is 4. We first count the equilateral triangles where all edges are drawn in the picture provided. There are 16 such triangles with side length 1 (10 pointing up, 6 pointing down), 7 with side length 2 (6 pointing up, 1 pointing down), 3 with side length 3 (all pointing up) and 1 with side length 4 (pointing up). This gives a total of 27 equilateral triangles so far.

   There are 6 triangles with side length $\sqrt{3}$. Three of them are shown in the below picture, and the other three are obtained from these by rotating the below diagram by 120 degrees.



There are two remaining triangles, shown in the below diagram.



This gives the total of 35 equilateral triangles.

3. Note that 1 is not composite. The smallest possible prime divisor of a non-obviously composite number is 7. If a non-obviously composite number has at least three prime factors (with multiplicity), it is therefore at least $7^3 = 343 > 200$. So every non-obviously composite number less than or equal to a product p_1p_2 of two prime numbers. Without loss of generality assume $p_1 \le p_2$.

   If $p_1 = 7$, as $p_1p_2 \le 200$, $p_2 \le 200/7 < 29$. So p_2 is either 7, 13, 17, 19 or 23, yielding 5 choices (note we must exclude 11).

   If $p_1 = 13$, then $p_2 \le 200/13 < 17$ so $p_2 = 13$. So there is one choice.

   $p_1 \ge 17$ is not possible as then $p_2 \ge 17$ as well and $p_1p_2 \ge 17^2 > 200$, a contradiction.

   So there are only 6 non-obviously composite numbers less than 200. Therefore the probability of one being chosen at random is $6/200 = 3/100$.

4. If X is a nonempty set of people, we write $\mu(X)$ for the average age of the people in X. We repeatedly use the following fact: If A and B are two disjoint sets of people, then either $\mu(A) \le \mu(A \cup B) \le \mu(B)$ or $\mu(B) \le \mu(A \cup B) \le \mu(A)$.

   (a) Suppose somebody whose age is between 30 and 40 moves from Willow Springs to Meadowbank. Since this person is older than the average age in Meadowbank, the average age in Meadowbank will increase. Since this person is younger than the average age in Willow Springs, the average age in Willow Springs will also increase, as required.

   (b) Suppose person X moves from Willow Springs to Meadowbank, and then person Y moves from Meadowbank to Willow Springs, so that the average age of both towns increases with both moves. In order for the average age of Willow Springs to increase with the first move, X must be at most 40 years old. Since the initial average age of Meadowbank is 30, this will cause the new average age of Meadowbank to be less than 40. When Y leaves Meadowbank, the average age of Meadowbank increases, so Y is less than 40 years old. But the average age in Willow Springs is at least 40, so when Y arrives in Willow Springs, the average age will decrease. This is a contradiction, so the premise is impossible.

5. Let $b = |AB|$ and let h be the distance between the lines AB and CD. Then the area of the parallelogram $ABCD$ is bh. Therefore

$$bh = 2024.$$

Triangle AED has base AE of length $\frac{1}{4}b$ and height h so its area is

$$\frac{1}{2}\left(\frac{1}{4}b\right)h = \frac{bh}{8} = \frac{2024}{8} = 253.$$

Triangle BEF has base BE of length $\frac{3}{4}b$ and height $\frac{h}{2}$ (as F is the midpoint of BC). Therefore its area is

$$\frac{1}{2}\left(\frac{3}{4}b\right)\left(\frac{h}{2}\right) = \frac{3}{16}2024 = \frac{759}{2}.$$

Therefore the area of $CDEF$ is

$$\begin{aligned}
\text{Area}(CDEF) &= \text{Area}(ABCD) - \text{Area}(AED) - \text{Area}(BEF) \\
&= 2024 - 253 - \frac{759}{2} \\
&= \frac{2783}{2}.
\end{aligned}$$

6. First consider the case where Jack puts both black balls in the same bucket. Then Jill will always win whenever she chooses a bucket that does not contain a black ball. So to maximise Jill's chances of winning we need to maximise the chances that Jill wins when she picks the bucket with both black balls. This probability is maximised when the number of white balls in this bucket is maximised. Since no bucket can be empty, Jack maximises the number of white balls in this box by placing one white ball in each of the $b-1$ buckets without a black ball, and the remaining $w-(b-1)$ white balls in the bucket containing both black balls. In this case, the probability of Jill winning is

$$\frac{b-1}{b} + \frac{1}{b}\frac{w-b+1}{w-b+3} = 1 - \frac{2}{b(w-b+3)}.$$

Now consider the case where Jack puts both black balls in different buckets. Suppose there are a white balls in one of these buckets and b white balls in the other. Note that since no bucket can be empty, $a \leq w-b+2$ and $b \leq w-b+2$.

The probability of Jill losing is therefore equal to

$$\frac{1}{b}\left(\frac{1}{a+1} + \frac{1}{b+1}\right) > \frac{2}{b(w-b+3)}.$$

Therefore the optimal strategy for Jack is to put one white ball in each of $b-1$ buckets, and every remaining ball in the other bucket. The probability of Jill winning in this case is

$$1 - \frac{2}{b(w-b+3)}$$