The University of Melbourne School Mathematics Competition, 2024

SENIOR DIVISION

1. A rectangle inscribed in a circle has area equal to half the area of a square inscribed in the same circle. What is the ratio of the two side lengths of the rectangle?

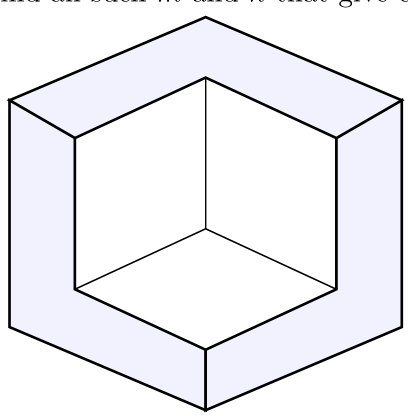
2. In the city Neves, the year is written using base 7 in reverse order, although it still agrees with the year in the rest of the world. For example this year in Neves is written

$$N1265 = 1 \times 1 + 2 \times 7 + 6 \times 7^2 + 5 \times 7^3 = 2024$$

and $N0002 = 0 \times 1 + 0 \times 7 + 0 \times 7^2 + 2 \times 7^3 = 686$. When next will the year written in both Neves notation and decimal notation agree?

3. Choose three points on a circle independently with uniform probability. What is the probability that the triangle with vertices given by the three points has an angle greater than ninety degrees?

4. From a cube of integer side length n, a smaller cube of integer side length m has been removed from one corner, as in the picture, so that as integers, the volume of the resulting shape is greater than its surface area by 3 units. Find all such m and n that give cubes with this property.



5. Over a ten week term, each student in a class of 25 students had to give two presentations to the class on a topic chosen by the teacher. Each day, one student was randomly chosen to give their presentation, so that after 5 weeks everyone had presented once. For the second half of the term, again the students were randomly chosen to give their presentations, although one student was missing due to illness and did not present. A student complained that they had less than a week between presentations, while others had more than four weeks between presentations. For each of the students who gave two presentations, the teacher counted the number of days between their two presentations and added these together to get the total number of days between presentations for the whole class. What is the maximum number of total days the teacher could have calculated?

6. On the square integer grid consisting of all integer pairs (m, n) with $\max\{|m|, |n|\} \leq 2024$, there is a real valued function F satisfying the following conditions:

(1) if $\max\{|m|, |n|\} = 2024$, then

$$F(m, n) = \frac{mn}{1 + |mn|}$$

(2) if $\max\{|m|, |n|\} < 2024$, then

$$F(m, n) = \frac{1}{4}\Big(F(m+1, n) + F(m-1, n) + F(m, n+1) + F(m, n-1)\Big).$$

Prove that $F(m, n) > 0$ whenever $m > 0$ and $n > 0$.