The University of Melbourne–School of Mathematics and Statistics
School Mathematics Competition, 2023

INTERMEDIATE DIVISION SOLUTIONS

1. Let m be a positive integer, such that $2m+1$ is a perfect square. Show that $m+1$ is the sum of two successive perfect squares.

Solution: Suppose $2m + 1 = n^2$, with n an integer. Since n^2 is odd, so is n. Write $n = 2k + 1$, where k is an integer. Then $2m + 1 = (2k + 1)^2$, which can be solved to give

$$m = \frac{4k^2 + 4k}{2} = 2k^2 + 2k,$$

so

$$m + 1 = 2k^2 + 2k + 1 = k^2 + (k + 1)^2.$$

2. Consider the arithmetic progression 1, 4, 7, 10, ..., 100. Let A be any set of twenty distinct numbers chosen from this arithmetic progression. Prove that there must be two distinct numbers in A that sum to 104.

Solution: There are 34 numbers in this arithmetic progression. There are 16 possible pairs whose sum is 104, notably $100 + 4,\ 97 + 7,\ \cdots, 55 + 49$.). If you retain 20 distinct numbers, you are eliminating 14. So at most you can eliminate 14 of the 16 possible pairs, leaving at least 2 pairs.

3. Two vertical posts, one of height h_1 and the other of height h_2 stand on level ground, a distance l apart. A supporting wire is stretched from the top of each pole to the base of the other. Find the height above ground h_3 of the point at which the two wires cross.

Solution. Let x be the distance from the foot of the pole of height h_2 to the point on the ground directly below where the wires cross. The distance from this point to the foot of the pole of height h_1 is therefore $l - x$. From similar triangles, we see that $\frac{h_3}{l-x} = \frac{h_2}{l}$, and

$\frac{h_3}{x} = \frac{h_1}{l}$. Eliminating x and solving for h_3, one obtains

$$h_3 = \frac{h_1 h_2}{h_1 + h_2}.$$

4. If you ask n people what their birthdate is (ignoring the year), what is the smallest number of people you must ask to have at least a 50% chance of finding someone with the same birthdate as you? You can ignore leap years, so assume 365 days in a year. You can express your result as the solution of an equation, without needing to solve that equation.

Solution: If $N = 365$ is the number of days in the year, the chance that somebody's birthdate misses matching yours is $(N-1)/N$. When you ask n people their birthdates, the chances that none of them have your birthday is $[(N-1)/N]^n$. The solution to the problem is thus given by the solution of the equation

$$\frac{1}{2} = 1 - \left(\frac{N-1}{N}\right)^n,$$

with $N = 365$. The solution is $n = 252.6519\ldots$, so 253 rounded to the nearest integer.

5. Let a, b, c, d be positive integers satisfying $ab + cd = 34$, $ac + bd = 46$ and $ad + bc = 31$. Find $a + b + c + d$. Then find all solution sets (a, b, c, d).

Solution: Add the last two equations, this gives $(a + b)(c + d) = 77$. So $(a + b)$, $(c + d)$ must be 7 and 11 in some order. So $a + b + c + d = 7 + 11 = 18$.
Now if $a+b = 7$, and we consider all possible pairs (a, b), it is only the pairs $(1, 6),\ (6, 1)$ and $(2, 5),\ (5, 2)$ that give a value for cd such that $c + d = 11$, being the pairs$(4, 7),\ (7, 4)$, and $(3, 8),\ (8, 3)$ respectively. Substituting into the given equations in the problem statement, one finds that the only solutions are $(a, b, c, d) = (1, 6, 4, 7),\ (6, 1, 7, 4),\ (4, 7, 1, 6),\ (7, 4, 6, 1)$, and $(a, b, c, d) = (2, 5, 3, 8),\ (5, 2, 8, 3),\ (3, 8, 2, 5),\ (8, 3, 5, 2)$.

6. From a point P on the circumference of a circle of radius r, a tangent line is drawn to a point T, where the length of the line PT is 10 units. The shortest distance from T to the circumference of the circle is 5 units. A straight line is drawn from T cutting the circle at point X and then further at point Y. The length of TX is 7.5 units. Find (a) The radius r of the circle, and (b) the length XY.

Solution: Let O be the centre of the circle, and Z be the point such that XZ is a diameter. Then triangle OPT is a right triangle, so $OT^2 = r^2 + PT^2$, and $OT = r + 5$, so $r^2 + 100 = (5 + r)^2$, so $r = 7.5$. Now consider the triangles OYT and OYX. Let x be the length XY, and α be the angle $\angle XTO$. From the triangle XTO, $\cos\alpha = \frac{12.5^2 + 7.5^2 - 7.5^2}{25 \cdot 7.5} = \frac{12.5}{15}$. From the triangle YTO, then $\cos\alpha = \frac{12.5^2 + (x+7.5)^2 - 7.5^2}{25 \cdot (x+7.5)} = \frac{12.5}{15}$, which has solutions $x = 0, 17.5/3$, so the length xy is $17.5/3 = 5.833333\cdots$.

7. Find all real solutions of the equation

$$x^3 + 4x^2 + 8x + \frac{1}{x^3} + \frac{4}{x^2} + \frac{8}{x} = 140.$$

Solution: Rearrange as

$$\left(x^3 + \frac{1}{x^3}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 8\left(x + \frac{1}{x}\right) = 140.$$

Rewrite as

$$\left(x + \frac{1}{x}\right)^3 + 4\left(x + \frac{1}{x}\right)^2 + 5\left(x + \frac{1}{x}\right) = 148.$$

Set $y = x + \frac{1}{x}$, to yield

$$y^3 + 4y^2 + 5y - 148 = 0.$$

Since $148 = 1 \cdot 4 \cdot 37$ we first look for a solution involving these integers (positive or negative). It's easy to see that one root is $y = 4$, so

$$y^3 + 4y^2 + 5y - 148 = (y - 4)(y^2 + 8y + 37).$$

Notice that the discriminant of the quadratic is negative $(64 - 148 = -84)$, so $y = 4$ is the only real solution. So $x + \frac{1}{x} = 4$, or

$$x^2 - 4x + 1 = 0,$$

from which we obtain the solutions $x = 2 \pm \sqrt{3}$.