The University of Melbourne School Mathematics Competition, 2022

JUNIOR DIVISION

Solutions

1. You arrive at Melbourne at 2:14pm. You pass every Echuca bound train that leaves Melbourne before 2:14pm and arrives at Echuca after 11am. A train arrives at Echuca before 11am if and only if it departs Melbourne after 7:46am. So you pass every train that departs Melbourne between 7:46am and 2:14pm. As the trains depart every hour on the hour, the trains you pass are the departures at 8am, 9am, 10am, 11am, 12pm, 1pm and 2pm. Therefore there are 7 Echuca bound trains that you pass before you reach Melbourne.

2. A number has no prime factors greater than 6 if and only if it is of the form $2^a 3^b 5^c$. To be at most 100, we must have $c \leq 2$.

   If $c = 2$, there are four options 25, 50, 75 and 100.

   Now suppose $c = 1$.

   If $b = 0$ then $2^a \leq 20$ so $a \leq 4$ and there are 5 choices.
   If $b = 1$ then $2^a \leq 20/3$ so $a \leq 2$ and there are 3 choices.
   If $b = 2$ then $2^a \leq 20/9$ so $a \leq 1$ and there are 2 choices.
   If $b \geq 3$ then $2^a \leq 20/27$ which has no solutions.
   Therefore there are $5 + 3 + 2 = 10$ choices with $c = 1$.

   Now suppose $c = 0$.

   If $b = 0$ then $2^a \leq 100$ so $a \leq 6$ and there are 7 choices.
   If $b = 1$ then $2^a \leq 100/3$ so $a \leq 5$ and there are 6 choices.
   If $b = 2$ then $2^a \leq 100/9$ so $a \leq 3$ and there are 4 choices.
   If $b = 3$ then $2^a \leq 100/27$ so $a \leq 1$ and there are 2 choices.
   If $b = 4$ then $2^a \leq 100/81$ so $a = 0$ and there is one choice.
   If $b \geq 5$ then $3^b > 100$ so there are no further choices
   Therefore there are $7 + 6 + 4 + 2 + 1 = 20$ choices with $c = 0$.

   Therefore there are in total $4 + 10 + 20 = 34$ numbers between 1 and 100 inclusive which have no prime factors greater than 6.

   So the probability of picking such a number is 17/50.

3. After n turns have been taken, the total number of coins taken is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

   assuming the coins have not yet run out. For $n = 63$ this sum is 2016. Therefore after 63 turns there are 6 coins remaining, which are all taken by Wilfred (as 63 is divisible by 3, so they will have all taken an equal number of turns, so it will be Wilfred's turn next). So the number of coins Wilfred gets is

$$1 + 4 + 7 + \cdots + 58 + 61 + 6,$$

the number of coins Chantelle gets is

$$2 + 5 + 8 + \cdots + 59 + 62,$$

and the number of coins Thelma gets is

$$3 + 6 + 9 + \cdots + 60 + 63.$$

Thelma gets more coins than Chantelle because she has the same number of turns as Chantelle and gets one more coin each turn.

Thelma gets 21 turns, and on each turn gets two more coins than Wilfred did on his turn. This gives Thelma a lead of 42 coins over Wilfred before Wilfred takes the last six. As $6 < 42$, Thelma gest more coins than Wilfred.

Therefore Thelma ends up with the most coins.

4. Let t, c and m be the numbers of trucks, cars and motorbikes respectively that Sean buys. We have two equations

$$t + c + m = 22,$$

$$9t + c + m/2 = 44.$$

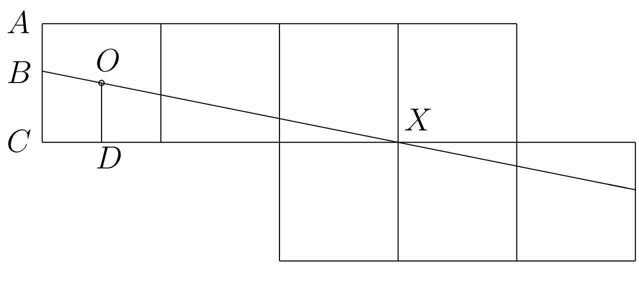
Taking twice the second equation minus the first equation gives

$$17t + c = 66.$$

As all the variables are non-negative integers, it must be that $0 \leq c \leq 22$. Rearranging the third equation shows that $66 - c$ is divisible by 17. The only integer c between 0 and 22 for which $66 - c$ is divisible by 17 is $c = 15$.

Having established $c = 15$ we can substitute it into the last equation to see that $t = 3$ and substitute these values in to either of the first equations to see that $m = 4$. As there is a unique solution in non-negative integers, Sally can work out exactly how many of each type of toy Sean bought.

5. We refer to the following picture



Ignoring the leftmost square, the remaining six squares are symmetric under rotation by 180 degrees. Therefore the amount of cake under the cut in those six squares is equal to the amount of cake above the cut in those six squares. Since the cut must cut the cake in half, it must therefore also cut the leftmost square in half. In order to do this, it must pass through the centre of the leftmost square.

Let O be the centre of the leftmost square. Let C be the corner of the cake as depicted in the diagram and let D be the foot of the perpendicular from O to CX.

The triangles XBC and XOD are similar triangles with similarity ratio $|XC|/|XD| = 30/25 = 6/5$. Therefore $|BC|/|OD| = 6/5$. As O is the centre of the square $|OD| = 5$cm and hence $|BC| = 6$cm. Therefore

$$|AB| = |AC| - |BC| = 4\text{cm}.$$

6. There are two solutions: $(m, n) = (1, 2)$ and $(m, n) = (3, 3)$.

   If $m = 2$ then $m! + 3 = 5$ which is not a perfect square

   If $m = 4$ then $m! + 3 = 27$ which is not a perfect square.

   Now if $m \geq 5$, then $m!$ is divisible by ten. Therefore $m! + 3$ has its last digit (in decimal notation) a 3. The only possible last digits of squares are 0,1,4,5,6 and 9. Therefore $m! + 3$ is not a perfect square for $m \geq 5$.

   This exhausts all cases except the two where we have solutions.

7. Let the square have side length 1. The area of an equilateral triangle with side length x is $\sqrt{3}x^2/4$. To have area equal to 1 requires $x = 2/\sqrt[4]{3}$. Note that this is larger than $\sqrt{2}$, which is the length of the diagonal of the original square.

   So if the square is divided into three triangles and rearranged to form an equilateral triangle, each side of the resultant equilateral triangle must be made up of two (or more) sides of the three composite triangles. But it is impossible to divide a triangle into three triangles where each of the sides of the original triangle is divided. Therefore Pinocchio is lying.

   On the other hand, it is possible to create some triangle. For example: