The University of Melbourne School Mathematics Competition, 2022

SENIOR DIVISION
Solutions

1. Find all non-negative integers m and n satisfying:

$$2^m + 2^{m+1} + 2^{m+2} + 2^{m+3} = 3^n + 3^{n+1} + 3^{n+2} + 3^{n+3}.$$

Solution:

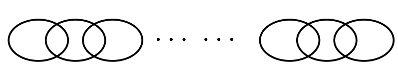
$$\begin{aligned} 2^m \cdot 3 \cdot 5 = 2^m \cdot 15 &= 2^m(1 + 2 + 2^2 + 2^3) = 2^m + 2^{m+1} + 2^{m+2} + 2^{m+3} \\ &= 3^n + 3^{n+1} + 3^{n+2} + 3^{n+3} = 3^n(1 + 3 + 3^2 + 3^3) = 3^n \cdot 40 = 2^3 \cdot 3^n \cdot 5. \end{aligned}$$

Hence $m = 3$ and $n = 1$ (due to unique factorisation by primes).

2. A swimmer is swimming against the current of a river. She loses her sandal which is strapped to her back and only realises this after swimming for a further 10 minutes. On realising, she immediately turns around to retrieve the sandal which has been flowing with the current of the river. The swimmer swims with constant effort in both directions and the river flows at a constant rate. If she reaches the sandal after it has moved 1000 metres down river from where she lost it, how fast is the river flowing?

Solution: The time taken by the swimmer to swim away from and back to the flowing sandal is independent of the flow since it moves the swimmer and sandal equally. Hence it takes equal time to return to the sandal since we may as well assume the flow is zero. Hence the swimmer takes 10 minutes to return to the sandal after turning around. The sandal has traveled 1000 metres in for 20 minutes so the river is flowing at 50 metres per minute.

3. A chain consists of 2022 rings linked end-to-end in a line. On each ring is a (possibly zero) number of scratches. There are 2570 total scratches on the rings of the chain. In any block of 11 adjacent rings there are 14 scratches in total. How many total scratches are in the middle two rings of the chain?



Solution: The 1st ring must have the same number of scratches as the 12th ring and 23rd ring, and so on, in order to preserve 14 scratches in blocks of 11 adjacent rings. Similarly, rings n and $n + 11$ have the same number of scratches. The first $2013 = 11 \times 183$ rings contain $2562 = 14 \times 183$ scratches, while rings 2014 to 2022 contain the remaining 8 scratches. Hence rings 2012 and 2013 contain $14 - 8 = 6$ scratches. But this equals the number of scratches in the middle two rings 1010 and 1011 since $2012 = 1010 + 11 \times 92$ and $2013 = 1011 + 11 \times 92$. The middle two rings of the chain contain 6 scratches.

4. Fill in the missing digits for the following long division. (The first digit of any number is not zero.)

$$
\begin{array}{cccccc|ccccccccc}
 & & & & & & & & & & & \ast & \ast & 8 & \ast \\ \cline{7-15}
\ast & \ast & \ast & \ast & 8 & \ast & \ast & 8 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & & & \\ \cline{7-12}
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & \ast & & \\
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & \ast & & \\ \cline{7-13}
\multicolumn{6}{c}{} & & & \ast & 8 & \ast & \ast & \ast & \ast & \\
\multicolumn{6}{c}{} & & & \ast & 8 & \ast & \ast & \ast & \ast & \\ \cline{9-14}
\multicolumn{6}{c}{} & & & \ast & \ast & \ast & \ast & \ast & 8 & \ast \\
\multicolumn{6}{c}{} & & & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \cline{9-15}
\end{array}
$$

Solution: We see that $8\times$ the six digit divisor has six digits. Hence in order that $m\times$ the six digit divisor has seven digits for $m < 10$ we must have $m = 9$ so the appearance of seven digit terms twice in the calculation implies the quotient is necessarily of the form $\ast 989$. The third last line must be of the form $88\ast\ast\ast\ast$ and the fourth last line must be of the form $98\ast\ast\ast\ast$ since $8\times$ six digit number > 800000 and it cannot be > 900000 since the difference of the two is ≥ 100000. Hence, the six digit divisor is necessarily of the form $11118\ast$ so that $9\times$ divisor has seven digits and since $111280 \times 8 = 890240$ violates $8\times$ divisor $= 88\ast\ast\ast\ast$. This produces:

$$
\begin{array}{cccccc|ccccccccc}
 & & & & & & & & & & & \ast & 9 & 8 & 9 \\ \cline{7-15}
1 & 1 & 1 & 1 & 8 & \ast & \ast & 8 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & & & \\ \cline{7-12}
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & \ast & & \\
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & \ast & & \\ \cline{7-13}
\multicolumn{6}{c}{} & & & 9 & 8 & \ast & \ast & \ast & \ast & \\
\multicolumn{6}{c}{} & & & 8 & 8 & \ast & \ast & \ast & \ast & \\ \cline{9-14}
\multicolumn{6}{c}{} & & & 1 & \ast & \ast & \ast & \ast & 8 & \ast \\
\multicolumn{6}{c}{} & & & 1 & \ast & \ast & \ast & \ast & 8 & \ast \\ \cline{9-15}
\end{array}
$$

$9 \times 111180 + a = 1000620 + 9a$ has second last digit 8 hence $a = 7$. Also $8 \times 111187 = 889496$ and we have:

$$
\begin{array}{cccccc|ccccccccc}
 & & & & & & & & & & & \ast & 9 & 8 & 9 \\ \cline{7-15}
1 & 1 & 1 & 1 & 8 & 7 & \ast & 8 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & & & \\ \cline{7-12}
\multicolumn{6}{c}{} & \ast & \ast & \ast & \ast & \ast & \ast & \ast & & \\
\multicolumn{6}{c}{} & 1 & 0 & 0 & 0 & 6 & 8 & 3 & & \\ \cline{7-13}
\multicolumn{6}{c}{} & & & 9 & 8 & \ast & \ast & \ast & \ast & \\
\multicolumn{6}{c}{} & & & 8 & 8 & 9 & 4 & 9 & 6 & \\ \cline{9-14}
\multicolumn{6}{c}{} & & & 1 & 0 & 0 & 0 & 6 & 8 & 3 \\
\multicolumn{6}{c}{} & & & 1 & 0 & 0 & 0 & 6 & 8 & 3 \\ \cline{9-15}
\end{array}
$$

Add 11 to the quotient $a989$ to get $b000$ for $b = a + 1$. Then

$$a989 \times 111187 = (b000 - 11) \times 111187 = bb * * * * * * *$$

Hence the first digit of the quotient is 7 to get 8 as the second digit of the product.

$$\begin{array}{r|ccccccccc}
 & & & & & & 7 & 9 & 8 & 9 \\ \hline
1\ 1\ 1\ 1\ 8\ 7 & 8 & 8 & 8 & 2 & 7 & 2 & 9 & 4 & 3 \\
 & 7 & 7 & 8 & 3 & 0 & 9 & & & \\ \cline{2-7}
 & 1 & 0 & 9 & 9 & 6 & 3 & 9 & & \\
 & 1 & 0 & 0 & 0 & 6 & 8 & 3 & & \\ \cline{2-8}
 & & & 9 & 8 & 9 & 5 & 6 & 4 & \\
 & & & 8 & 8 & 9 & 4 & 9 & 6 & \\ \cline{4-9}
 & & & 1 & 0 & 0 & 0 & 6 & 8 & 3 \\
 & & & 1 & 0 & 0 & 0 & 6 & 8 & 3 \\ \cline{4-10}
\end{array}$$

5. Prove that there are no real solutions to the following:

$$\pm\sqrt{x} \pm \sqrt{x+1} \pm \sqrt{x+2} \pm \sqrt{x+3} = 0.$$

Note that different choices of ± 1 give 16 different equations.

Solution:

$$\begin{aligned}
&\pm\sqrt{x} \pm \sqrt{x+1} \pm \sqrt{x+2} \pm \sqrt{x+3} = 0 \\
\Rightarrow\quad & \pm\sqrt{x} \pm \sqrt{x+2} = \pm\sqrt{x+1} \pm \sqrt{x+3} \\
\Rightarrow\quad & x + x + 2 \pm 2\sqrt{x(x+2)} = x + 1 + x + 3 \pm 2\sqrt{(x+1)(x+3)} \\
\Rightarrow\quad & \sqrt{(x+1)(x+3)} = \pm 1 \pm \sqrt{x(x+2)} \\
\Rightarrow\quad & (x+1)(x+3) = x(x+2) + 1 \pm 2\sqrt{x(x+2)} \\
\Rightarrow\quad & x + 1 = \pm\sqrt{x(x+2)} \\
\Rightarrow\quad & (x+1)^2 = x(x+2) \\
\Rightarrow\quad & 1 = 0
\end{aligned}$$

which is a contradiction hence there are no solutions.

6. Frida and Zac grow and sell trees. They plant a field full of trees. In the first year, three trees die and they sell one fifth of the remaining trees, the largest ones, and keep the other four-fifths. In the second year, two trees die and they sell one seventh of the remaining trees, the largest ones, and keep the other six-sevenths. In the third year, one tree dies and they sell one thirteenth of the remaining trees, the largest ones, and keep the other twelve-thirteenths. In the fourth year, three trees die and they sell one fifth of the remaining trees, the largest ones, and keep the other four-fifths. What is the minimum number of trees they could have planted?

Solution: Any two solutions must differ by $5 \times 7 \times 13 \times 5 = 2275$ because the remainder is taken from only one of the solutions, requiring $\frac{4}{5} \times \frac{6}{7} \times \frac{12}{13} \times \frac{4}{5}$ times the difference to be an integer. Begin with m trees. Suppose after the first year m trees remain. Then:

$$\tfrac{4}{5}(m-3) = m \Rightarrow m = -12$$

Try the same for the second year:

$$\tfrac{6}{7}(m-2) = m \Rightarrow m = -12$$

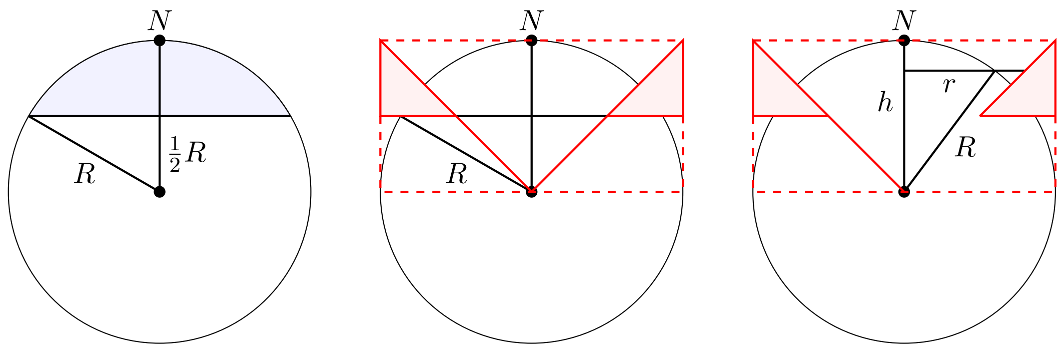
And for the third year:

$$\tfrac{12}{13}(m-1) = m \Rightarrow m = -12$$

The fourth year is the same as the first. So we see that -12 trees allows the divisions with the appropriate remainders. But we need a positive number of trees, hence $-12 + 2275 = 2263$ allows the divisions with the appropriate remainders. This is the smallest such solution since any other solution must differ by at least 2275.

7. What is the probability that a point chosen randomly inside a ball is closer to the north pole, i.e. the top most point of the ball, than to the centre of the ball?

Solution 1: The probability is given by the volume of the blue shaded region in the sphere, obtained by rotating the circle below around a vertical axis to achieve a sphere, divided by the volume of the sphere.



The volume of the blue shaded region is equal to the volume of the solid circular shaped red shaded region obtained by the same rotation of the circle around a vertical axis. This fact uses the theorem that two regions in 3-space that lie between two parallel planes have equal volumes if the areas of their cross-section intersections with each plane parallel to these two planes are equal.

The area of any disk blue cross-section is πr^2, for r shown in the third picture and the area of the annulus red cross-section at the same level is

$$\pi R^2 - \pi h^2 = \pi(R^2 - h^2) = \pi r^2$$

hence the blue and red volumes coincide.

$$\begin{aligned} \text{volume of red region } &= \text{volume of solid cylinder - volume of large cone + volume of small cone} \\ &= \pi R^2 \times \tfrac{1}{2}R - \tfrac{1}{3}\pi R^2 \times R + \tfrac{1}{3}\pi(\tfrac{1}{2}R)^2 \times \tfrac{1}{2}R \\ &= \frac{5}{24}\pi R^3 \end{aligned}$$

The volume of the sphere is $\frac{4}{3}\pi R^3$ ($= 2(\pi R^2 \times R - \frac{1}{3}\pi R^2 \times R)$ calculated in the same way). Hence the probability that a point chosen randomly inside a ball is closer to the north pole than to the centre of the ball is

$$\frac{\frac{5}{24}\pi R^3}{\frac{4}{3}\pi R^3} = \frac{5}{32}.$$

Solution 2: Using calculus, the volume of the blue shaded region is obtained by integrating the area πr^2 at height h where $h^2 + r^2 = R^2$ from $h = \frac{1}{2}R$ to $h = R$

$$\int_{\frac{1}{2}R}^{R} \pi r^2 dh = \int_{\frac{1}{2}R}^{R} \pi(R^2 - h^2)dh = \pi(R^2 h - \tfrac{1}{3}h^3)\big|_{\frac{1}{2}R}^{R} = \pi\left[R^3 - \tfrac{1}{3}R^3 - (\tfrac{1}{2}R^3 - \frac{1}{24}R^3)\right] = \frac{5}{24}\pi R^3$$

and the probability is $\frac{5}{32}$ as calculated above.