**The University of Melbourne–School of Mathematics and Statistics**
**School Mathematics Competition, 2022**

# INTERMEDIATE DIVISION - SOLUTIONS

1. Recall that

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.$$

How many zeros are at the end of the number 50!?

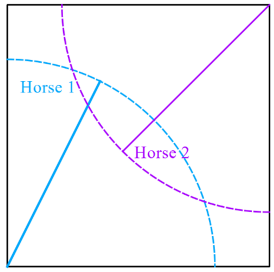
Solution: One obtains a zero by multiplying by a number ending in a zero, or by a number ending in a 5 provided there is a factor of 2 in that number. So in the product, the first 10 numbers $1, 2, \cdots 10$ contribute two zeros, one from 10 and one from 5 (times 2). Similar reasoning for the numbers 11 to 20 gives two zeros, one from 20 and one from 15 (times 14). From 21 to 30 we have three zeros, coming from 30 and 25, as 25 contains two factors of 5, so together with 24 contributes two zeros.

Arguing similarly for each lot of 10 numbers, we get two zeros from $31-40$, and we get three zeros from $41-50$.

Thus we have three ranges of numbers contributing two zeros and two ranges contributing three zeros, for a total of 12 zeros.

2. A square field contains two horses, attached by ropes to diagonally opposite corners of the field. Both ropes are of equal length, such that each horse can reach half the field. Which area is larger, the area of the field that both horses can reach, or the area that neither horse can reach?

Solution: Each horse can access half the area of the field. A diagonal line from the north-west to the south-east corner of the square field divides it in two. Above the diagonal line is an area equal to half the shared area that both horses can access. Below the diagonal line is an area equal to half the area that neither horse can access. Given that the accessible area is fixed at half the total area, these two afore-mentioned areas must be equal.




3. Calculate

$$\sin^2(1) + \sin^2(2) + \sin^2(3) + \cdots + \sin^2(90),$$

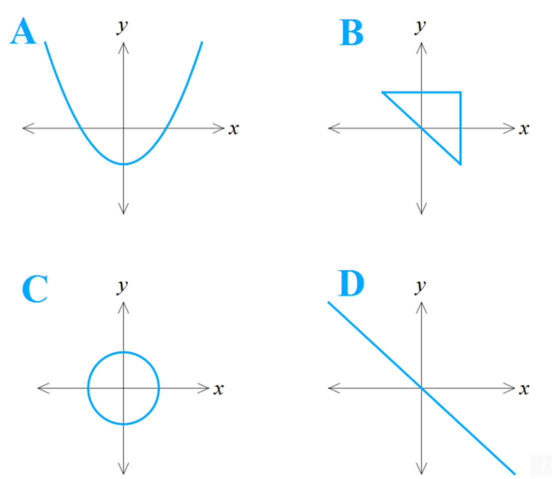
where the arguments are in degrees.

Solution. Recall that $\sin(\theta) = \cos(90 - \theta)$, so this sum is equal to

$$\cos^2(89) + \cos^2(88) + \cos^2(87) + \cdots + \cos^2(0).$$

So adding the sum of sines and the sum of cosines gives twice the answer. Using the identity $\cos^2\theta + \sin^2\theta = 1$, the total of the two sums is 89+1+1, so the required sum is 45.5.

4. Which of the four following graph(s) could satisfy the relation $p(x) + p(y) = 0$, for some polynomial function $p$? (Recall that a polynomial can be a constant, or linear, as in $a + bx$, or quadratic, as in $ax^2 + bx + c$, or cubic etc.)



Solution: If $p(x) + p(y) = 0$, the equation, and hence the graph, must be symmetric in $x$ and $y$. This means that, apart from a possible translation of the co-ordinate axes, the figure is symmetric with respect to the line $y = x$. Figures B, C and D have this property. However, we can rule out B as follows: Part of B is a horizontal line segment, say at $y = c$. So there are infinitely many solutions to the equation $p(x) + p(c) = 0$. So $p(x) + p(c)$ is a polynomial with infinitely many roots, so is zero. So $p$ must be constant, which doesn't describe B.

We still have to demonstrate that C and D can be realised. If $p(x) = ax + b$, then $p(x) + p(y) = 0$ gives the equation of a straight line. In the case of fig. D the straight line passes through the origin, so $p(x) = ax$.

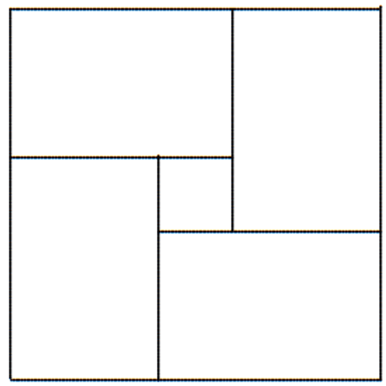
If $p(x) = ax^2 + bx + c$, then $p(x) + p(y) = 0$ gives the equation of a circle. If the circle is centred at the origin, then $p(x) = ax^2 + c$.

Comment: The interpretation of the question was intended to be that the graphs were *defined by* p(x) + p(y) = 0 rather than *satisfied* p(x) + p(y) = 0. The latter interpretation allows the trivial solution p(x) = 0 for all four graphs, and this solution was awarded partial marks.

5. Suppose you wish to send a carefully chosen gift to each of eight friends. You wrap the gifts in parcels, and address them to the recipients. However, you become distracted, and end up forgetting which gift is in which parcel, so you address them at random, and hope that you are correct. What is the probability that only 1 is correctly addressed?

Solution: You can correctly address one of eight parcels, so you have eight choices. You must then incorrectly address the remaining seven parcels. So of the 7! possible permu-tations of the remaining seven parcels, you must choose those permutations with none of the addresses correct. Such permutations are known as *derangements*, and these were first studied by Euler, who proved both recurrences $a(n) = (n-1)(a(n-1) + a(n-2))$ and $a(n) = na(n-1) + (-1)^n$. With two parcels, either both are correctly labelled, or both are incorrectly labelled, so the number of derangements of 1 element is zero. So $a(1) = 0$, and from the second recurrence we find the sequence 0, 1, 2, 9, 44, 265, 1854, etc. so the number of ways of addressing only 1 of the 8 parcels correctly is $8 \times 1854 = 14832$, so the required probability is $14832/8! = 103/280$.

6. Four rectangles are arranged in a square pattern so that they surround a smaller square, as shown in the figure. Let $O$ be the area of the outer square, and let $I$ be the area of the inner square. If $O/I = 9 + 4\sqrt{5}$, determine the ratios of the lengths of the sides of the rectangles.



Solution: Let the rectangles be of dimension $x$ times $y$, with $x > y$. Then the side-length of the outer square is $x + y$, and that of the inner square is $x - y$. So

$$\frac{(x+y)^2}{(x-y)^2} = 9 + 4\sqrt{5} = (2+\sqrt{5})^2.$$

So

$$x + y = (2+\sqrt{5})x - (2+\sqrt{5})y.$$

So

$$x + \sqrt{5}x = 3y + \sqrt{5}y,$$

and

$$\frac{x}{y} = \frac{\sqrt{5}+1}{2}.$$

Note that this is the golden ratio.

7. Find all the decimal numbers of the form 101, 10101, 1010101, etc (i.e. alternating 1s and 0s, starting and ending with a 1) that are prime numbers?

Solution: Let $N = 1010\cdots101$, with $k$ occurrences of 1, for some $k \geq 2$. Then $99N = 10^{2k} - 1 = (10^k+1)(10^k-1)$. So if $N$ is prime, it divides either $10^k+1$ or $10^k-1$. So one of $\frac{99}{10^k-1} = \frac{10^k+1}{N}$ and $\frac{99}{10^k+1} = \frac{10^k-1}{N}$ is an integer. For $k > 2$ both $10^k+1$ and $10^k-1$ are greater than 99, so this gives a contradiction. So only $k = 2$ is a possibility, and indeed $N = 101$ is the only prime of this form.