The University of Melbourne School Mathematics Competition, 2022

SENIOR DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The* ***seven*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

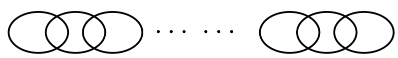
Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

1. Find all non-negative integers m and n satisfying:

$$2^m + 2^{m+1} + 2^{m+2} + 2^{m+3} = 3^n + 3^{n+1} + 3^{n+2} + 3^{n+3}.$$

2. A swimmer is swimming against the current of a river. She loses her sandal which is strapped to her back and only realises this after swimming for a further 10 minutes. On realising, she immediately turns around to retrieve the sandal which has been flowing with the current of the river. The swimmer swims with constant effort in both directions and the river flows at a constant rate. If she reaches the sandal after it has moved 1000 metres down river from where she lost it, how fast is the river flowing?

3. A chain consists of 2022 rings linked end-to-end in a line. On each ring is a (possibly zero) number of scratches. There are 2570 total scratches on the rings of the chain. In any block of 11 adjacent rings there are 14 scratches in total. How many total scratches are in the middle two rings of the chain?



4. Fill in the missing digits for the following long division. (The first digit of any number is not zero.)

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                              *  *  8  *
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5. Prove that there are no real solutions to the following:

$$\pm\sqrt{x} \pm \sqrt{x+1} \pm \sqrt{x+2} \pm \sqrt{x+3} = 0.$$

Note that different choices of ± 1 give 16 different equations.

6. Frida and Zac grow and sell trees. They plant a field full of trees. In the first year, three trees die and they sell one fifth of the remaining trees, the largest ones, and keep the other four-fifths. In the second year, two trees die and they sell one seventh of the remaining trees, the largest ones, and keep the other six-sevenths. In the third year, one tree dies and they sell one thirteenth of the remaining trees, the largest ones, and keep the other twelve-thirteenths. In the fourth year, three trees die and they sell one fifth of the remaining trees, the largest ones, and keep the other four-fifths. What is the minimum number of trees they could have planted?

7. What is the probability that a point chosen randomly inside a ball is closer to the north pole, i.e. the top most point of the ball, than to the centre of the ball?