The University of Melbourne School Mathematics Competition, 2021

SENIOR DIVISION
Solutions

1. Place the digits $\{1, 2, 3, 4, 5, 6, 7, 7, 8\}$ in the 3×3 square to form six 3-digit numbers that add to 2021. It is enough to find one solution. An example adding to 2723 instead of 2021 is given below.

7	2	3
4	5	6
1	7	8

$\mapsto 723 + 456 + 178 + 741 + 257 + 368 = 2723$

Solution:

a	b	d
c	f	g
e	h	i

The sum is

$$S = 200a + 110(b + c) + 101(d + e) + 20f + 11(g + h) + 2i.$$

The minimum value of S occurs when the smallest digits are assigned to the largest coefficients, i.e.

$$(a, b, c, d, e, f, g, h, i) = (1, 2, 3, 4, 5, 6, 7, 7, 8) \Rightarrow S = 1949$$

hence $S \geq 1949$. If 1 is swapped from a to another digit, the increase will be at least $200 - 110 = 90$ but $1949 + 90 > 2021$ hence $a = 1$. Similarly, if one of $\{2, 3, 4, 5\}$ is swapped from $\{b, c, d, e\}$, the increase will be at least $101 - 20 = 81$ but $1949 + 81 > 2021$ hence $\{b, c, d, e\} = \{2, 3, 4, 5\}$, as a set and we haven't determined the order.

The strategy is to begin with the minimal solution and swap entries to increase the sum S by $2021 - 1949 = 72$. Any swap will increase the sum by a multiple of 9. Swap $2 \leftrightarrow 4$ and $3 \leftrightarrow 5$ to increase S by $4 \times 9 = 36$ and swap $6 \leftrightarrow 8$ to increase S further by $2 \times 18 = 36$. This yields:

1	4	2
5	8	7
3	7	6

$\mapsto 142 + 587 + 376 + 153 + 487 + 276 = 2021$

(One can show that there are exactly 4 solutions by applying $2 \leftrightarrow 3$ and/or $4 \leftrightarrow 5$ to the solution above.)

2. Prove that there are only finitely many triples of positive integers (a, b, c) satisfying

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution: By symmetry we may assume $a \le b \le c$.

We have $3 \le a \le 6$ since if $a \le 2$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > \frac{1}{2}$ and if $a \ge 7$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{3}{7} < \frac{1}{2}$.

Consider the four cases $a = 3, 4, 5, 6$.

$\mathbf{a = 3}$. Then $b \ge 7$ (else $\frac{1}{a} + \frac{1}{b} \ge \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$) hence

$$\frac{1}{c} \ge \frac{1}{2} - \frac{1}{3} - \frac{1}{7} = \frac{1}{42} \qquad \Rightarrow c \le 42.$$

$\mathbf{a = 4}$. Then $b \ge 5$ (else $\frac{1}{a} + \frac{1}{b} \ge \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$) hence

$$\frac{1}{c} \ge \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \frac{1}{20} \qquad \Rightarrow c \le 20.$$

$\mathbf{a = 5}$. Then $b \ge 5$ (else a is not the smallest) hence

$$\frac{1}{c} \ge \frac{1}{2} - \frac{1}{5} - \frac{1}{5} = \frac{1}{10} \qquad \Rightarrow c \le 10.$$

$\mathbf{a = 6}$. Then $b \ge 6$ (else a is not the smallest) hence

$$\frac{1}{c} \ge \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{6} \qquad \Rightarrow c = 6.$$

Hence

$$3 \le a \le b \le c \le 42$$

so there are at most $40 \times 40 \times 40 = 64000$ solutions, where a is not necessarily smallest, which is finite.
Note: There are exactly 46 solutions.

Solution 2: Suppose for the sake of contradiction there are infinitely many solutions (a, b, c). Then without loss of generality a achieves infinitely many distinct values implying

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{2} - \frac{1}{a}$$

achieves infinitely many distinct values (since $\frac{1}{2} - \frac{1}{x}$ is injective on the positive integers). So one of b and c must also achieve infinitely many distinct values. Thus two of a, b, c achieve infinitely many distinct values (we won't assume which two). Now without loss of generality $a \le b \le c$ and thus $3 \le a \le 6$. Then

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{1}{3} + \frac{2}{b}$$

so $b \le 12$. This contradicts two of a, b, c achieving infinitely many distinct values.

3. Let $O(n)$ be the number of odd coefficients of $(1 + x)^n$. For example, $O(3) = 4$ since $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$ has 4 odd coefficients. Calculate

$$\sum_{n=1}^{511} O(n).$$

Solution: Draw Pascal's triangle modulo 2, which is also known as Sierpinski's triangle. The top two rows give a triangle of three 1s so that

$$\sum_{n=0}^{1} p(n) = 3$$

where $p(0) = 1$. The triangle of three 1s appears 3 times in the first four rows so that

$$\sum_{n=0}^{3} p(n) = 3 \times 3 = 3^2.$$

The first four rows appears three times in the first eight rows to give:

$$\sum_{n=0}^{7} p(n) = 3^2 \times 3 = 3^3.$$

Similarly, as the number of rows is doubled, the previous pattern appears three times, and the number of 1s is tripled, giving:

$$\sum_{n=0}^{2^k-1} p(n) = 3^k.$$

Hence

$$\sum_{n=1}^{511} p(n) = 3^9 - 1 = 19682.$$

4. Let p be a polynomial of degree 2020 satisfying

$$p(1) = 1, \quad p(2) = 2, \quad p(3) = 3, \quad ..., \quad p(2020) = 2020, \quad p(2021) = 0.$$

Prove that for any integer n, $p(n)$ is an integer.

Solution: The polynomial

$$p(x) = x - 2021 \frac{(x-1)...(x-2020)}{2020!}$$

satisfies $p(n) = n$ for $n = 1, ..., 2020$ and $p(2021) = 0$. Moreover $p(x)$ is necessarily given by this formula since any other polynomial $q(x)$ must satisfy $p(n) - q(n) = 0$ for $n = 1, ..., 2021$ and degree $p(x) - q(x) = 2020$. But then $p(x) - q(x) \equiv 0$ since any such polynomial has a factor of $(x-1)(x-2)...(x-2021)$ which is of degree 2021.

For any integer n, we can equivalently write

$$p(n) = \begin{cases} n - 2021\binom{n-1}{2020} & n > 0 \\ \\ n - 2021\binom{2020-n}{2020} & n \leq 0 \end{cases}$$

which clearly takes integer values.

5. Ruby, Sam and Theo are each given one of three consecutive positive integers (for example 24, 25 and 26). They know their own number and that the three numbers are consecutive and positive, but do not know the numbers given to the others. The following sequence of true statements is made, in order:

Ruby says: "I do not know all three numbers."
Sam says: "I do not know all three numbers."
Theo says: "I do not know all three numbers."
Ruby says: "I do not know all three numbers."
Sam says: "I now know all three numbers."
Theo says: "I do not know all three numbers."

What number is Theo given?

Solution: The question was originally misstated, omitting the fact that the integers are known to be positive.

A. If anyone was given 1 then that would person would know from the beginning that the other two were given 2 and 3. Since each person first says "I do not know the other numbers" no-one was given 1.

B. If Theo was given 2 then he would know the other numbers are 3 and 4 since by **A** he would know the others do not have 1. But he says "I do not know the other numbers" so we (and the other players) may conclude that Theo's number is greater than 2.

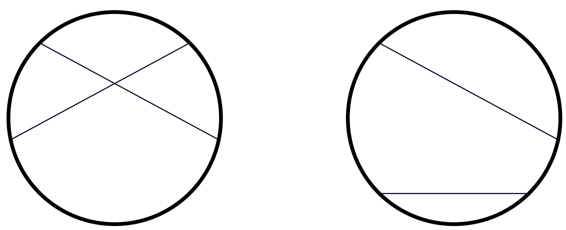
C. Similarly to **B**, if Ruby was given 2 then she would know the other numbers are 3 and 4 just before the second time she speaks. But she says for a second time "I do not know the other numbers" so we (and the other players) may conclude that Ruby's number is greater than 2.

D. If Sam was given 2 or 3 then just before the second time she speaks she would know the numbers of the others: 3 and 4 if she was given 2, or 4 and 5 if she was given 3, since she knows that the others are given numbers greater than 2. If Sam was given a number greater than 3 then she would not know the others' numbers. So we (and the other players) may conclude that Sam was given 2 or 3.

E. By **D**, Theo was given 3, 4 or 5. If Theo was given 3 then just before the second time he speaks he knows that Sam was given 2 and Ruby was given 4. If Theo was given 5 then just before the second time he speaks he knows that Sam was given 3 and Ruby was given 4. Since he says for a second time "I do not know the other numbers" we may conclude that Theo was given 4.

In conclusion, (Ruby, Sam, Theo) were given $(2, 3, 4)$ or $(3, 5, 4)$ so Theo was given 4.

6. Choose two chords of a circle independently and randomly by choosing their endpoints on the circle with uniform probability. What is the probability that the two chords intersect? The two chords intersect in the picture on the left and do not intersect in the picture on the right.



Solution 1: We may assume the circle has circumference 1 and write the circle as the unit interval $[0, 1]$, where 0 and 1 are identified/glued together to produce the circle. Two chords are determined by a point in $[0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ where the first two factors determine the first chord and the final two factors determine the second chord. The probability space is in fact

$$\frac{[0,1] \times [0,1]}{\mathbb{Z}_2} \times \frac{[0,1] \times [0,1]}{\mathbb{Z}_2}$$

since the same chord is produced when the endpoints are flipped. We can ignore this since every pair of chords appears 4 times via flipping, and will not affect the proportion = probability of chords that intersect.

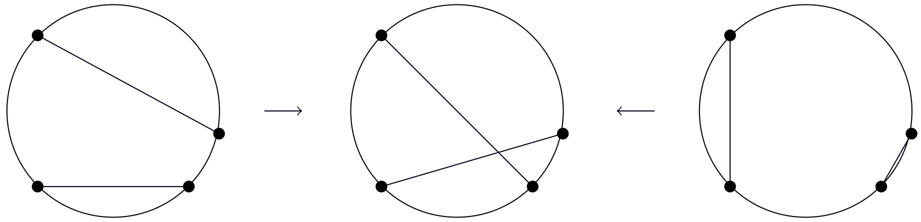
Let

$$M \subset [0,1] \times [0,1] \times [0,1] \times [0,1], \quad D \subset [0,1] \times [0,1] \times [0,1] \times [0,1]$$

be the subset of those chords that intersect, respectively don't intersect, so their union gives the entire space $M \cup D = [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ i.e. $Pr(M) + Pr(D) = 1$. There is a map

$$D \to M$$

defined by using the same endpoints to construct chords that intersect as in the diagram.



This map is $2:1$ hence $Pr(D) = 2Pr(M)$ and since $Pr(M) + Pr(D) = 1$ we find that the probability that the two chords intersect is $Pr(M) = 1/3$.

Solution 2: By rotation, we may assume that the first endpoint is at $0 \in [0,1]$, and say the second endpoint is at $x \in [0,1]$. Then the second chord does not intersect the first chord if both of its endpoints lie in $(0,x)$ or both in $(x,1)$. The probability of the first is $x \times x$ and the probability of the second $(1-x)\times(1-x)$ hence the probability that they don't intersect is

$$x^2 + (1-x)^2.$$

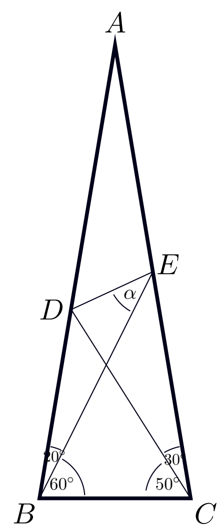
In order to "add' all such probabilities we need to integrate to get the probability that the two chords don't intersect:

$$Pr(D) = \int_0^1 (x^2 + (1-x)^2)dx = \frac{x^3}{3} - \frac{(1-x)^3}{3}\bigg|_0^1 = \frac{2}{3}.$$

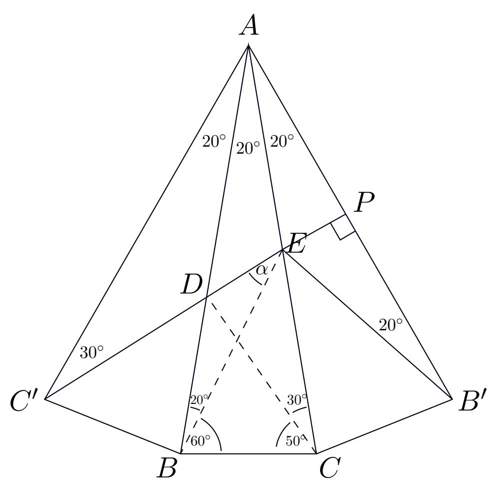
Hence

$$Pr(M) = 1 - Pr(D) = 1 - \frac{2}{3} = \frac{1}{3}.$$

7. Find the angle $\angle BED = \alpha$ shown in the following diagram.



Solution 1. Reflect triangle ABC twice as follows: reflect C in AB to get C' and reflect B in AC to get B' as in the diagram. Drop a perpendicular from C' to AB' at P on AB'.



Then P divides AB' into equal parts since $AB'C'$ is equilateral. Hence by symmetry

$$\angle EB'A = 20$$

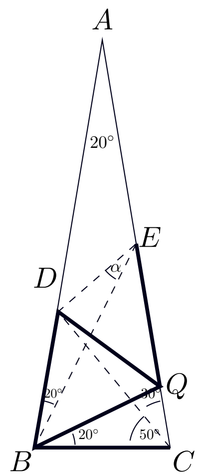
as indicated. Then $\angle BC'A = 30$ and $\angle EB'A = 20$ imply that $CD =$ reflection of $C'D$ and $BE =$ reflection of $B'E$ are the two lines in the original diagram with angles 50 and 60 i.e.

$$50 = \angle BC'D = \angle BCD, \quad \text{and} \quad 60 = \angle CB'E = \angle CBE.$$

The important point is that the perpendicular $C'P$ passes through D and E. Finally,

$$\alpha = \angle AEP - \angle BEC = 70 - 40 = 30.$$

Solution 2.
Construct Q on AC such that $\angle QBC = 20$ as in the diagram.



Since $\angle BCD = \angle BDC = 50$, we have $BC = BD$. Also $\angle BCQ = \angle BQC = 80$, so $BQ = BC = BD$ and $\angle DBQ = 60$, so triangle DBQ is equilateral, and $DQ = EQ$. Next $\angle EBQ = \angle BEQ = 40$, so triangle QEB is isosceles hence $EQ = BQ = DQ$. The lines

$$BC = BD = EQ = BQ = DQ$$

are thickened. This means triangle QDE is also isosceles. Then

$$2(\alpha + 40) + 40 = 180$$

and thus $\alpha = 30$.

Solution 3. Trigonometric proof.
Put $AB = 1 = AC$, $BC = x = BD$, $AE = y = BE$.
Cosine rule on $AEB \Rightarrow y = \frac{1}{2\cos(20)}$.
Sine rule on $BCE \Rightarrow \frac{x}{\sin(40)} = \frac{y}{sin(80)} = \frac{y}{2\sin(40)\cos(40)} \Rightarrow x = \frac{y}{(2\cos(40))} = \frac{y}{1+y}$ $(*)$
where $(*)$uses

$$2\cos(40) = 1 + y \Leftarrow 8\cos(20)^3 - 6\cos(20) - 1 = 0 \Leftarrow 8\cos(20)^3 - 6\cos(20) = 2\cos(60) = 1.$$

Now, $x = \frac{y}{1+y} \Rightarrow y = \frac{x}{1-x} \Rightarrow BED$ and ACD are similar triangles.
$\Rightarrow \alpha = 30$.