The University of Melbourne School Mathematics Competition, 2021

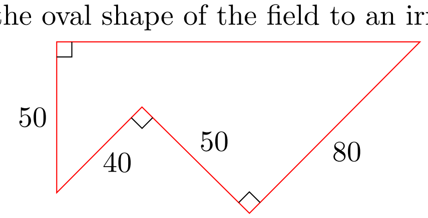
JUNIOR DIVISION

Solutions

1. Not hard enough. You are competing in the ABC's game show "Hard Quiz", presented by comedian Tom Gleeson. With your topic of choice, *Mathematics*, you have reached the final head-to-head round and are competing for the limited edition Big Brass Mug. Tom likes to think he is smarter than the competitors on his show and with a particularly smug face asks you the following: "n comedians performed at this year's Melbourne Comedy Festival. No two comedians told the same number of jokes, no comedian told 2021 jokes, and no comedian told more than n or no jokes at all. What is the largest number of comedians that could have performed at the comedy festival?" Answer Tom's question, and wipe that smile off his face.

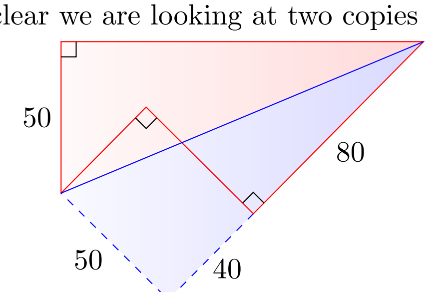
Solution:
Since no comedian told 2021 jokes or no jokes, 2021 is the largest possible number: Tom G. telling one joke, Will A. telling two jokes, Judith L. telling three jokes, ..., Hannah G. telling 2020 jokes. If there had been more than 2020 comedians, then, since no one told 2021 jokes, by the pigeonhole principle two comedians would have to tell the same number of jokes.

2. Now that's cricket. As a consequence of the COVID-19 pandemic and the three-month Greater Melbourne lockdown, fewer people are currently physically active than before the pandemic struck. A special task-force of the state government led by Minister for Community Sport, Ros Spence, is currently looking at initiatives to get the population active again on the sporting field. One idea of the task-force aimed at boosting the number of year 7 and 8 students playing cricket is to make the game more exciting by changing the oval shape of the field to an irregular pentagon:



with dimensions measured in metres. What is the total area of the proposed new playing field?

Solution:
By moving two of the sides it is clear we are looking at two copies of the same right-sided triangle:



Hence the area of the field is $50 \times (80 + 40) - 50 \times 40 = 50 \times 80 = 4000$, or $4000\,\mathrm{m}^2$ to be precise.

3. Our proud COVID-19 record. Australia has been widely praised around the world for dealing with the COVID-19 crisis much better than most countries. However, this is not our only great COVID-19 feat. We have also topped the world-rankings in hoarding toilet paper during the crisis, and our toilet-paper wars have been receiving as much attention around the world as our low infection rates. At your local supermarket, n greedy customers buy up all the stock of toilet paper. The first customer buys 10 packs of toilet paper, plus one tenth of the remaining stock. The next customer buys 20 packs of toilet paper plus one tenth of the remaining stock. This continues until the nth customer buys $10 \times n$ packs of toilet paper plus one tenth of the remaining stock. The shelves are now completely empty. If all n greedy customers bought the same amount of toilet paper, how many packs of toilet paper did your local supermarket stock?

Solution:
The nth customer bought $10 \times n$ plus one tenth of the remaining stock, to completely leave the shelves empty. Hence that one tenth must have been one tenth of nothing and they bought $10 \times n$ packs of toilet paper. The second-last customer bought $10 \times (n-1)$ packs plus one tenth of the remaining stock. That one tenth must thus have been

$$\frac{10}{9} \times n$$

so that nine tenth was exactly the $9 \times (10/9) \times n = 10 \times n$ bought by the nth customer. We thus have

$$10 \times (n-1) + \frac{10}{9} \times n = 10 \times n$$

which is solved for $n = 9$. Each greedy customer thus bought 90 packs of toilet paper so that the supermarket must have stocked $90 \times 9 = 810$ packs of toilet paper.

Solution 2:
Let x be the number of packs. The first customer buys $10 + (x-10)/10$, and the second customer buys $20 + [x - 10 - (x-10)/10 - 20]/10$. We have

$$\begin{aligned} & 10 + (x-10)/10 = 20 + [x - 10 - (x-10)/10 - 20]/10 \\ \Rightarrow \quad & (x-10)(1/10 - 1/10 + 1/100) = 20 - 2 - 10 = 8 \\ \Rightarrow \quad & x = 810. \end{aligned}$$

4. The devil to pay. A group of 64 people is assigned the numbers 1 to 64 by the number devil. Nobody in the group knows which number has been assigned to them or to anyone else in the group. The task of the 64 is to find out who has been assigned the numbers 1, 2 and 3. Any group of up to 8 people can ask the number devil to point out which person in that group has the lowest number, second lowest and third lowest number. The number devil will not, however, reveal what their actual numbers are. If a group of n people asks this question of the number devil it will cost them n dollars. In the best case scenario, what is the minimum total cost to the group of 64 to find out which person has been given the number 1, which person has been given the number 2 and which person has been given the number 3? In the worst case scenario, what is the minimum total cost to the group of 64 to find out which person has been given the number 1, which person has been given the number 2 and which person has been given the number 3?

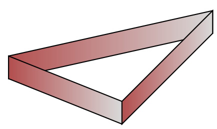
Solution: Divide the 64 people into 8 groups, denoted by the letters $A, B, \ldots, H$, and let each group of 8 ask the number devil to point out who has the lowest, second-lowest and third-lowest number in their group. This comes at a total cost of $8 \times 8 = 64$ dollars. Call the person in group A with the lowest, second-lowest and third-lowest number, A_1, A_2 and A_3, respectively. Do the same for the other 7 groups. Anyone in their group who is not among the lowest three cannot possibly be among the lowest three overall, so that we are left with the 24 people $A_1, A_2, A_3, \ldots, H_1, H_2, H_3$. Note that $A_2, B_2, \ldots, H_2$ can at best have the second-lowest number overall, and $A_3, B_3, \ldots, H_3$ can at best have the third-lowest number overall.
In the best case scenario 1,2 and 3 can be determined for only 8 more dollars, for a total of 72 dollars. For example, if $A_1 = 1$, $A_2 = 2$ and $A_3 = 3$, then it would suffice to form the group (of eight) given by $A_3, B_1, C_1, \ldots, H_1$ and ask the number devil to point out the lowest three numbers in that group.

Once it is clear that A_1 is the lowest it is all done. This approach, however, is a dangerous gamble because because if it fails it may cost more than 77 dollars, which is the maximum that one needs to pay. Not taking risks by assuming any fortunate scenarios leads to a cost of at most 77 dollars as follows. Form the group (of eight) given by $A_1, B_1, \dots, H_1$ and ask the number devil to point out the lowest, second-lowest and third-lowest among them. This again brings to sub-total to 72 dollars. Assume (without loss of generality) that A_1 is the lowest, B_1 the second lowest and C_1 the third lowest. This implies that $A_1 = 1$, only A_2 or B_1 can be 2 and one of A_2, A_3, B_1, B_2, C_1 must be 3. Obviously, A_3 could only be 3 if $A_2 = 2$, B_2 could only be 3 if $B_1 = 2$, and C_1 could only be 3 if $B_2 = 2$. More specifically, the five remaining scenarios are

$$(A_1, A_2, A_3) = (1, 2, 3), \quad (A_1, A_2, B_1) = (1, 2, 3), \quad (A_1, B_1, A_2) = (1, 2, 3),$$
$$(A_1, B_1, B_2) = (1, 2, 3), \quad (A_1, B_1, C_1) = (1, 2, 3).$$

Forming a final group (of five) A_2, A_3, B_1, B_2, C_1 and paying the number devil a final 5 dollars will now be enough. It is not possible to do better. You could first ask the number devil to order A_2 and B_1. If $A_2 < B_1$, then it suffices to finally let the number devil order A_3 and B_1 (for a total of 76 dollars). However, if $B_1 < A_2$, you still need to determine the order among A_2, B_2 and C_1, again requiring a total of 77 dollars.

5. Year of the ox. According to ancient Chinese folklore, oxen are hard workers, intelligent, reliable, and never demand praise. They are also the preferred prey of the Eurasian grey wolf. A Chinese oxherd has lost almost all of her oxen to wolves, and wishes to protect her few remaining oxen by building a 2-metre high triangular ox-pen as in the following diagram:



She has 2021 thin pieces of pre-cut timber, the smallest of size 200×1 (height times width, both measured in centimetres), the next smallest of size 200×2, the next smallest of size 200×3, and so on, with the largest of size 200×2021. She selects three pieces of timber to build the ox-pen. If the smallest piece has size 200×1001, in how many different ways can she select the remaining two pieces? You may use that $1 + 2 + 3 + \cdots + k = k(k + 1)/2$.

Solution:
Let $n < m < p$ be the three side-lengths of the pen. (We of course have $n = 1001$ but we ignore this for now). To form a non-degenerate triangle (i.e., one that does not have an area of 0) we must have $n + m > p$. If $n = 1$ this is impossible. If $n = 2$ then we must have $p = m + 1$. Hence (m, p) must be one of

$$(3, 4), (4, 5), \dots, (2020, 2021)$$

giving the oxherd $2020 - 2$ possibilities. If $n = 3$ then we must have $p = m + 1$ or $p = m + 2$. Hence (m, p) must be one of

$$(4, 5), \dots, (2020, 2021), \quad (4, 6), (5, 7), \dots (2019, 2021)$$

giving the oxherd $(2020-3)+(2019-3) = (2020-3)+(2020-4)$ possibilities. Hopefully the following pattern is now clear: For fixed n there are

$$\underbrace{(2020 - n) + (2020 - n - 1) + \cdots + (2020 - 2n + 2)}_{n-1 \text{ terms}}$$

possibilities. If $n = 1001$ this gives

$$\begin{aligned}(2020 - 1001) + (2020 - 1002) + \cdots + (2020 - 2000)& \\ = 1019 + 1018 + \cdots + 20& \\ = (1 + 2 + \cdots + 1019) - (1 + 2 + \cdots + 19)& \\ = 1019 \times 1020/2 - 19 \times 20/2& \\ = 1019 \times 510 - 19 \times 10& \\ = 519690 - 190 = 519500.&\end{aligned}$$

6. Escape room. You and three of your friends are locked in an escape room. As part of the game you have all been given a number from 1 to 4, such that no two of you have the same number. In an adjoining room are four boxes numbered 1 to 4. Hidden inside each box is a key in the shape of one of the numbers 1 to 4. The keys, no two of which are the same number, have been placed in the boxes at random. Without opening a box you cannot see which key it contains. One at the time you can enter the second room and open (and then close) three of the four boxes. If all four of you manage to open a box containing a key matching your own number (do not take the key out!) you can all escape. You are *only* allowed to talk before the first person enters the second room. Show that if the four of you are clever you have a better than 40% chance of escaping.

Solution: It is not a great idea to randomly open 3 boxes. There is a 75% chance (3 out of 4) that each one of you opens a box with the right number, so that a random approach would give you a $(3/4)^4 = 81/256$ chance (which is a bit less than 32%) of escaping. Better is for the player with number n to open the box labelled n. If the number inside is n they can stop. If it is not n but m, they should next open the box labelled m. If the number inside is n they can again stop. If it is not n but k, they should next open the box labelled k. If this contains n, great. If not, bad luck. Now how likely is it that all four of you are successful, and you can escape? Think of the numbers inside the boxes as a *permutation* of the numbers written on the outside. If for example we have

$$\begin{array}{rl} \text{outside:} & 1,\ 2,\ 3,\ 4 \\ \text{inside:} & 4,\ 1,\ 3,\ 2 \end{array}$$

it can be encoded diagrammatically as

$$\begin{tikzcd} & 1 \arrow[dr] & & \\ 2 \arrow[ur] & & 4 \arrow[ll] & 3 \arrow[loop right] \end{tikzcd}$$

since box one contains a 4, box 4 contains a 2 and box 2 contains a 1; box 3 contains a 3. We say that we have one 3-cycle and one 1-cycle. Similarly,

$$\begin{array}{rl} \text{outside:} & 1,\ 2,\ 3,\ 4 \\ \text{inside:} & 3,\ 1,\ 4,\ 2 \end{array}$$

is encoded diagrammatically as $\begin{tikzcd} 1 \arrow[r] & 3 \arrow[d] \\ 2 \arrow[u] & 4 \arrow[l] \end{tikzcd}$ which consists of one 4-cycle.

The proposed strategy fails if the permutation corresponding to the escape room is a 4-cycle, as in the second example. In the first example 1,2 and 4 will succeed after three attempts and 3 will succeed after just one attempt. For example player 1 will first find a 4, then a 3 and finally a 1. Of the $4 \times 3 \times 2 \times 1 = 24$ possible permutations of the numbers 1 to 4 we have: 6 permutations with one 4-cycle. Hence the strategy goes wrong with a probability of $8/24 = 1/4$ and goes right with a probability of $3/4$ which is much better that 40% of the time.

If you prefer not to think in terms of permutations, assume you adopt the above strategy. There are 24 different ways the numbers 1 to 4 can be distributed among the four numbered boxes. Those displayed in blue are good, those in red are bad:

outside:	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4
inside:	1, 2, 3, 4;	1, 2, 4, 3;	1, 3, 2, 4;	1, 3, 4, 2;	1, 4, 2, 3;	1, 4, 3, 2
outside:	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4
inside:	2, 1, 3, 4;	2, 1, 4, 3;	2, 3, 1, 4;	2, 3, 4, 1;	2, 4, 1, 3;	2, 4, 3, 1
outside:	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4
inside:	3, 1, 2, 4;	3, 1, 4, 2;	3, 2, 1, 4;	3, 2, 4, 1;	3, 4, 1, 2;	3, 4, 2, 1
outside:	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4;	1, 2, 3, 4
inside:	4, 1, 2, 3;	4, 1, 3, 2;	4, 2, 1, 3;	4, 2, 3, 1;	4, 3, 1, 2;	4, 3, 2, 1.