The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2021

SENIOR DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The* ***seven*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

1. Place the digits $\{1, 2, 3, 4, 5, 6, 7, 7, 8\}$ in the 3×3 square to form six 3-digit numbers that add to 2021. It is enough to find one solution. An example adding to 2723 instead of 2021 is given below.

7	2	3
4	5	6
1	7	8

$\mapsto 723 + 456 + 178 + 741 + 257 + 368 = 2723$

2. Prove that there are only finitely many triples of positive integers (a, b, c) satisfying

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

3. Let $O(n)$ be the number of odd coefficients of $(1 + x)^n$. For example, $O(3) = 4$ since $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$ has 4 odd coefficients. Calculate

$$\sum_{n=1}^{511} O(n).$$

4. Let p be a polynomial of degree 2020 satisfying

$$p(1) = 1, \quad p(2) = 2, \quad p(3) = 3, \quad ..., \quad p(2020) = 2020, \quad p(2021) = 0.$$

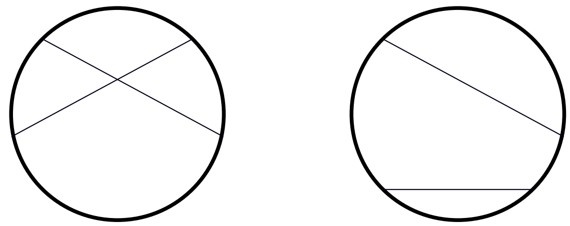
Prove that for any integer n, $p(n)$ is an integer.

5. Ruby, Sam and Theo are each given one of three consecutive integers (for example 24, 25 and 26). They know their own number and that the three numbers are consecutive, but do not know the numbers given to the others. The following sequence of true statements is made, in order:

Ruby says: "I do not know all three numbers."
Sam says: "I do not know all three numbers."
Theo says: "I do not know all three numbers."
Ruby says: "I do not know all three numbers."
Sam says: "I now know all three numbers."
Theo says: "I do not know all three numbers."

What number is Theo given?

6. Choose two chords of a circle independently and randomly by choosing their endpoints on the circle with uniform probability. What is the probability that the two chords intersect? The two chords intersect in the picture on the left and do not intersect in the picture on the right.



7. Find the angle $\angle BED = \alpha$ shown in the following diagram.