The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2021

INTERMEDIATE DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The* ***six*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

1. A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks and 40 black socks. I select socks one at a time from the drawer, but can't see the colour of the socks I select. What is the smallest number of socks I must select to ensure that I have at least ten pairs? (A pair means two socks of the same colour).

2. Twenty fair coins are simultaneously tossed. What is the probability that exactly 10 of them land heads? (Note: you do not need to evaluate your answer).

3. Show that if the integer $n \geq 6$ is composite (i.e. not a prime), then n divides $(n-1)!$ (Note that $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.)

4. Find all pairs of positive integers (m, n) such that $m^{1/m}$ is the square of $n^{1/n}$.

5. A triangle is drawn in the plane, such that the coordinates of the vertices are integers. The product of the lengths of two sides is also an integer, and, further, is a prime number. Similarly, the area is also a prime. Find the area of the triangle.

6. In triangle ABC, all angles are acute (less than 90 degrees). Point P is chosen arbitrarily on BC. Its reflection in AC is labelled T, and its reflection in AB is labelled S. Determine the position of point P so that the area of triangle ATS is a minimum.