THE UNIVERSITY OF MELBOURNE - BHP BILLITON SCHOOL MATHEMATICS COMPETITION 2006

INTERMEDIATE DIVISION

Time Allowed: THREE hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately.

The following suggestions are made for your guidance:

1. *The examiners will attach great weight to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant solution than for a clumsy solution.*

2. *The questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

3. *It may be necessary to spend considerable time on a problem before any real progress is made.*

4. *You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.*

5. *Credit will be given for partial solutions; however, a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks are **NOT** *allowed. Electronic calculators, tables, etc., may be used. Computers may not be used. Calculators capable of storing text should have their memories erased before use. Otherwise, normal examination conditions apply.*

Warning: *Make sure you have the correct problems (Senior, Intermediate or Junior) in front of you.*

1. Kim has just completed a 21 kilometre bike trip. If she had been able to ride 2 kilometres per hour faster, she would have completed her trip 15 minutes earlier. Find her speed.

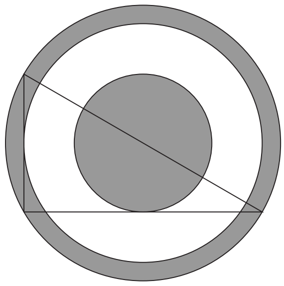
2. Five letters are written down from left to right satisfying

   (a) only the letters A, C, G and T are used; and

   (b) no neighbouring letters are the same.

   What is the probability that "CAT" will appear as three consecutive letters written from left to right?

3. In the following diagram, each of the three circles is centred at the midpoint of the longest side of the triangle. Prove that the two shaded regions have the same area.



4. In the 18th century, Euler proved the remarkable fact that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}.$$

   Use this to determine the value of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots.$$

5. Find all positive integers less than 10,000 which are equal to five times the product of their digits.

6. Recall that the absolute value of x is defined as

$$|x|=\begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

   Find all real values of x that satisfy

$$|x|-|x+2|+|x+4|-|x+6|+|x+8|=|x+1|-|x+3|+|x+5|-|x+7|+|x+9|.$$

7. Express the number $\frac{1}{2006}$ as the sum of the reciprocals of distinct integers of size less than 2006 — that is,

$$\frac{1}{2006}=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n},$$

   where $-2006 < a_i < 2006$ for all $i=1,2,\ldots,n$.