



THE UNIVERSITY OF
MELBOURNE

**The University of Melbourne–BHPBilliton
School Mathematics Competition
SENIOR DIVISION**

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately.

The following suggestions are made for your guidance.

- 1. Great weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*
- 2. The **seven** questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*
- 3. It may be necessary to spend considerable time on a problem before any real progress is made.*
- 4. You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.*
- 5. Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

*Textbooks are **NOT** allowed. Electronic calculators, tables, etc., may be used. Computers may not be used. Calculators capable of storing text should have their memories erased before use. Otherwise normal examination conditions apply.*

Candidates may attempt all questions.

Warning: *Make sure you have the correct problems (Senior, Intermediate or Junior) in front of you.*

1. Find all positive integer solutions of the equation $4m + 3n = 200$.
2. Two hospitals in the same town treat the same number of patients each month. The first hospital cures more patients each month than does the second. Each patient suffers from one of two possible diseases. It turns out that the second hospital cures a higher percentage of patients with disease 1, and also a higher percentage of patients with disease 2. Is this possible? (For your answer, you should either provide a numerical example consistent with the above observations, or prove that it is not possible).
3. Consider the sequence $1, 5, 6, 25, 26, 30, 31 \dots$ consisting of all positive integers which are either powers of 5, or sums of distinct powers of 5. (For example, $31 = 5^2 + 5^1 + 5^0$). Find the 50^{th} term in this sequence.
4. Consider paths in the plane with integer co-ordinates going from $(0, 0)$ to (N, N) . Each step in the path may be either one step in the positive x direction or one step in the positive y direction. At no stage in the path may any co-ordinate exceed N . (That is to say, paths are constrained to lie in the square with co-ordinates $(0, 0)$, $(0, N)$, $(N, 0)$ and (N, N)).
 - (a) Show that the number of different paths from $(0, 0)$ to (N, N) is $\binom{2N}{N}$.
 - (b) Using the answer to the previous part, or otherwise, find the total area (i.e. the number of squares) *below* all such paths. (For example, if $N = 1$, the answer is 1. If $N = 2$, the answer is 12, as of the six possible paths, there are 4 squares below one path ($yyxx$), three squares below one path ($yxyx$), two squares below each of two paths ($yxxy$ and $xyyx$), one square below one path ($xyxy$), and no squares below one path ($xyyy$).
5. You are given four points, lying in the plane, and told that the distance between any pair of points is not less than $\sqrt{2}$ and not greater than 2. Prove that these points must lie at the four corners of a square.
6. At a picnic, 15 friends decide to play rugby 7s. One of them is chosen to be the referee, and the others are split into two teams, each of 7 members. For balance, they decide that the two teams should have the same total weight. (Assume the weight of each player is an integer number of kilograms). They find that no matter who is chosen as the referee, this can always be done. Prove that all 15 friends must have the same weight.
7. Let x_1, x_2, \dots, x_n be positive real numbers, such that

$$\sum_{i=1}^n \frac{1}{n + x_i - 1} > 1.$$

Prove that $x_1 x_2 \dots x_n < 1$.