The University of Melbourne–School of Mathematics and Statistics
School Mathematics Competition, 2019

INTERMEDIATE DIVISION

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

1. *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

2. *The* ***seven*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

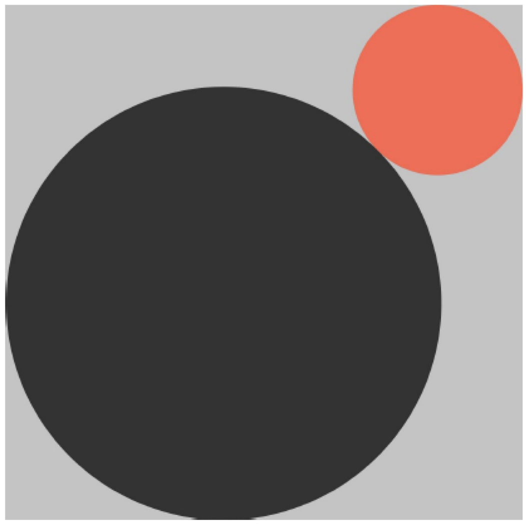
3. *It may be necessary to spend considerable time on a problem before any real progress is made.*

4. *You may need to do considerable rough work but you should then write out your final solution neatly, stating your arguments carefully.*

5. *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

1. Two circles are drawn inside a square, as shown in the picture. If the radius of the large circle is $3r$ and the radius of the small circle is r, find the area of the square.



2. Let a, b, c, d, e be positive integers satisfying $a < b < c < d < e$. The mean of the five integers is 10, and the median (this is the central value – there is an equal number of numbers smaller and larger than the median) is 7, and the difference between the smallest and largest number is 12. Find a, b, c, d, e.

3. It is a curious fact that

$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}.$$

Are there other such examples where $\sqrt{m+x} = m\sqrt{x}$ where m is a positive integer and x is real and positive? If so, find them.

4. Simplify the following:

$$\frac{(2^3-1)(3^3-1)\cdots(2019^3-1)}{(2^3+1)(3^3+1)\cdots(2019^3+1)}.$$

Express your answer as simply as possible, without spending time on large multiplications.

5. Am I more likely to get no more than two sixes when throwing 4 dice at once, or no more than one six when throwing 3 dice at once? (You must explain your answer, not

just say *the former* or *the latter*).

6. Let f be a function from positive integers to positive integers satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all positive integers n. Calculate $f(k)$ for $k = 1, \cdots, 10$.

7. The *digital sum* of a decimal integer n, written $\mathrm{DS}(n)$ is just the sum of the digits of n. So $\mathrm{DS}(345) = 3 + 4 + 5 = 12$.
Consider prime pairs $(p, p + \Delta)$ (so that both p and $p + \Delta$ are primes) such that $\mathrm{DS}(p(p + \Delta)) = \Delta$.
One example is $p = 2$, $\Delta = 5$, as $\mathrm{DS}(2(2 + 5)) = \mathrm{DS}(14) = 5$.
What possible values of $\Delta < 30$ can yield such prime pairs?