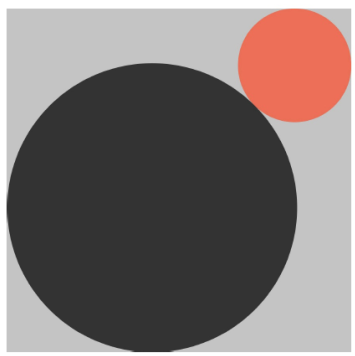
The University of Melbourne–School of Mathematics and Statistics
School Mathematics Competition, 2019

INTERMEDIATE DIVISION

1. Two circles are drawn inside a square, as shown in the picture. If the radius of the large circle is $3r$ and the radius of the small circle is r, find the area of the square.



Solution: Draw a diagonal passing through the centre of both circles. Label the top right corner of the square A, and the bottom left corner B, the centre of the smaller circle O_1 and that of the larger circle O_2. The diagonal passes through all these points.
AO_1 is $\sqrt{2}r$, BO_2 is $3\sqrt{2}r$, and O_1O_2 is $4r$. So AB is $4(r+\sqrt{2}r)$, and so the area is $1/2 \times AB^2 = 8r^2(3+2\sqrt{2})$.

2. Let a, b, c, d, e be positive integers satisfying $a < b < c < d < e$. The mean of the five integers is 10, and the median (this is the central value – there is an equal number of numbers smaller and larger than the median) is 7, and the difference between the smallest and largest number is 12. Find a, b, c, d, e.
Solution: Since the mean is 10, $a+b+c+d+e = 50$. Since the median is 7, $c = 7$. We are told $e - a = 12$. These three facts combine to give $2a + b + d = 31$. We want to find all solutions of this subject to the constraints $a < b < 7$, and $d < e = a + 12$. By

experimentation, $a = 5$, and $b = 6$ leads to $d = 15$ and $e = 17$. Any smaller value of a or b leads to no valid solution, and b cannot be larger.

3. It is a curious fact that

$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}.$$

Are there other such examples where $\sqrt{m+x} = m\sqrt{x}$ where m is a positive integer and x is real? If so, find them.
Solution: Squaring both sides gives $m + x = m^2x$, or $x = \frac{m}{m^2-1}$. Thus for all $m > 1$ we have examples of this phenomenon. The example given corresponds to $m = 3$.

4. Simplify the following:

$$\frac{(2^3-1)(3^3-1)\cdots(2019^3-1)}{(2^3+1)(3^3+1)\cdots(2019^3+1)}.$$

Express your answer as simply as possible, without spending time on large multiplications.
Solution: Use the fact that $n^3 - 1 = (n+1)(n^2+n+1)$ and $n^3+1 = (n+1)(n^2-n+1)$. So the product in the question becomes

$$\frac{1\cdot 7}{3\cdot 3}\frac{2\cdot 13}{4\cdot 7}\frac{3\cdot 21}{5\cdot 13}\frac{4\cdot 31}{6\cdot 21}\cdots\frac{2017\cdot(2018^2+2019)}{2019\cdot(2018^2-2017)}\frac{2018\cdot(2019^2+2020)}{2020\cdot(2019^2-2018)} =$$

$$\frac{1\cdot 2\cdot 3\cdots 2018\cdot(2019^2+2020)}{3\cdot 3\cdot 4\cdot 5\cdots 2019\cdot 2020} = 2\left(\frac{2019^2+2020}{3\cdot 2019\cdot 2020}\right) = \frac{4078381}{6117570}.$$

5. Am I more likely to get no more than two sixes when throwing 4 dice at once, or no more than one six when throwing 3 dice at once?
Solution: We need to compare the probability of throwing four dice and getting 0, 1 or 2 sixes to that of throwing three dice and getting 0 or 1 six. In the former case, the probability is

$$\left(\frac{5}{6}\right)^4 + 4\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)^3 + 6\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = \left(\frac{5}{6}\right)^2\left(\frac{51}{36}\right).$$

In the latter case, the probability is

$$\left(\frac{5}{6}\right)^3 + 3\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^2\left(\frac{48}{36}\right),$$

so the former probability is larger.
Alternative (better) solution given by Hadyn Tang (Trinity Grammar School) and Andres Buritica (Scotch College).

Model the situation as four consecutive dice rolls. Let A be the event that there are no more than 2 sixes in all four throws, and let B be the event that there is at most 1 six in the first three dice rolls. Consider an outcome $\omega \in B$. There is at most one six in the first three dice rolls, and the fourth roll can add at most one more six to the total number of sixes rolled. Therefore, there can be at most 2 sixes appearing on all four dice rolls, from which it follows that $\omega \in A$, so $B \subseteq A$. Additionally, note that the sequence of rolls $6, 6, 2, 2$ has a non-zero probability of occurring, and is an event in A but not in B. Accordingly, $B \subset A$ and $P(B) < P(A)$ (i.e. *the former* is more likely).

6. Let f be a function from positive integers to positive integers satisfying $f(n+1) > f(n)$ and$f(f(n)) = 3n$ for all positive integers n. Calculate $f(k)$ for $k = 1, \cdots, 10$.

Solution: To solve this type of problem, one tries to find solutions for $n = 0$, or $n = 1$ and/or some other property of f. Then one can conjecture more values, and more general results can usually be proved by induction. In this case, only low-order results are sought, so no inductive proof of general results is needed.
We start by showing $f(1) > 1$, as if $f(1) = 1$, we would have $3 = f(f(1)) = 1$, a contradiction.
More generally, $f(n) > n$, for if $f(n) \leq n$ for some $n = k$ we would have $f(k-1) < f(k) \leq k$, so $f(k-1) \leq k-1$. Repeat this argument $k-1$ times and we obtain $f(1) \leq 1$, in contradiction to our result above. Hence $f(n) > n$ for all n.
$f(1) = 2$. Suppose $f(1) = m \geq 3$. Then $3 = f(f(1)) = f(m) > m \geq 3$, a contradiction. So $f(1) = 2$.
So now from the given properties of f we can calculate several values:

$$f(2) = f(f(1)) = 3,\ f(3) = f(f(2)) = 6,\ f(6) = f(f(3)) = 9,\ f(9) = f(f(6)) = 18.$$

Given that $6 = f(3) < f(4) < f(5) < f(6) = 9$, it follows that $f(4) = 7,\ f(5) = 8$, so $f(7) = f(f(4)) = 12$ and $f(8) = f(f(5)) = 15$. Now $f(18) = f(f(9)) = 27$, and as there are 8 unknown values between $f(9) = 18$ and $f(18) = 27$ and 8 numbers between 18 and 27, it follows that $f(10) = 19,\ f(11) = 20,\ \cdots\ f(17) = 26$.

7. The *digital sum* of a decimal integer n, written $DS(n)$ is just the sum of the digits of n. So $DS(345) = 3 + 4 + 5 = 12$.

Consider prime pairs $(p, p + \Delta)$ (so that both p and $p + \Delta$ are primes) such that $DS(p(p + \Delta)) = \Delta$. One example is $p = 2$, $\Delta = 5$, as $DS(2(2 + 5)) = DS(14) = 5$.
What possible values of $\Delta < 30$ can yield such prime pairs?

Solution: Any natural number n can be written, in decimal form, as $n = \sum_k \alpha_k \cdot 10^k$. Its digital sum, $DS(n) = \sum_k \alpha_k$. Since $\alpha_k \cdot 10^k \equiv \alpha_k$, (mod(9), working in (mod 9) it follows that every number is equal to the sum of its digits. For the primes we are considering we require that $DS(p(p + \Delta)) = \Delta$. So $p(p + \Delta) - \Delta \equiv 0 \pmod 9$ or $(p - 1)(p + \Delta + 1) \equiv 8$

(mod 9). This excludes the values $p + \Delta \equiv 2,\ 5,\ 8 \pmod 9$, and since $p + \Delta$ is prime, the values $p + \Delta \equiv 3,\ 6\,,9$ are also excluded. This leaves $p + \Delta \equiv 1\,,4\,,7$ giving $p \equiv 5\,,8\,,2$ respectively. In each case we have $\Delta \ \equiv 5 \pmod 9$. If Δ is odd, the only solution is $p = 2$, $p + \Delta = 7$. If Δ is even the only solutions are $\Delta = 14 + 18k$, with $k = 0, 1, 2, 3, 4, \ldots$. So $\Delta = 14$ is the only solution less than 30.