The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2019

Time allowed: Three hours

These questions are designed to test your ability to analyse a problem and to express yourself clearly and accurately. The following suggestions are made for your guidance:

(1) *Considerable weight will be attached by the examiners to the method of presentation of a solution. Candidates should state as clearly as they can the reasoning by which they arrived at their results. In addition, more credit will be given for an elegant than for a clumsy solution.*

(2) *The* ***seven*** *questions are not of equal length or difficulty. Generally, the later questions are more difficult than the earlier questions.*

(3) *It may be necessary to spend considerable time on a problem before any real progress is made.*

(4) *You may need to do rough work but you should then write out your final solution neatly, stating your arguments carefully.*

(5) *Credit will be given for partial solutions; however a good answer to one question will normally gain you more credit than sketchy attempts at several questions.*

Textbooks, electronic calculators and computers are ***NOT*** *allowed. Otherwise normal examination conditions apply.*

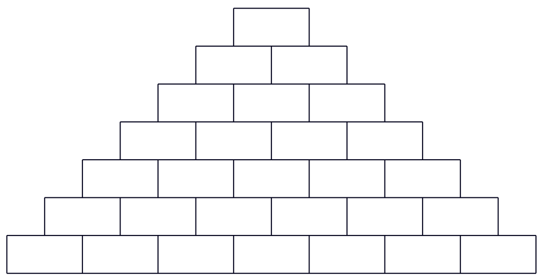
1. Let the positive integer N be given by

$$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{N}}}}} = 2019.$$

Simplify, as much as possible, the expression

$$\frac{\sqrt{N\sqrt{N\sqrt{N\sqrt{N\sqrt{N}}}}}}{N}.$$

2. Maya places integers in each of the boxes below, so that each integer is the sum of the two integers immediately below it. At most how many odd numbers can Maya write?



3. A class of 30 students plays the following game. The numbers 1 to 32 are written on the blackboard. The first student replaces two of the numbers on the blackboard with their sum decreased by 1; the second student replaces two of the numbers on the blackboard with their sum decreased by 2; and so on, so that the nth student replaces two of the numbers on the blackboard with their sum decreased by n. The game continues until each student has played. At the end of the game the final two numbers on the board are both positive. What are all the possible final two numbers on the board?

4. Find all products

$$N \times b_1b_2....b_{2019} \times b_1b_2....b_{2019} = b_1b_2....b_{2019}b_1b_2....b_{2019}$$

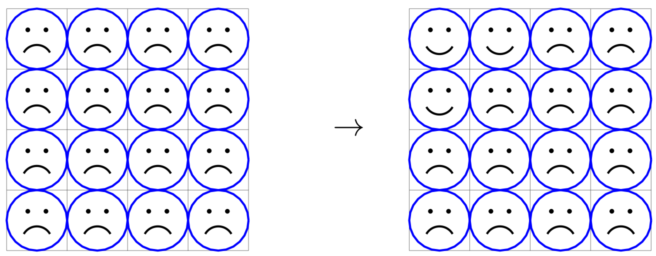
where N is an integer and each b_i is a digit, i.e. an integer satisfying $0 \leq b_i < 10$, $b_1 \neq 0$ and we write decimal numbers in terms of digits, so for example $b_1b_2 = 34$ if $b_1 = 3$ and $b_2 = 4$.

5. Given the longest length of a triangle, if we randomly choose its other two lengths, what is the probability that the triangle is acute? In other words, given a length $a > 0$, randomly choose $0 < b < a$ and $0 < c < a$ such that $b + c > a$, and form a triangle with side lengths a, b and c, then what is the probability that all angles of the triangle are less than $\pi/2$?

6. Consider a 4×4 grid with each position containing a happy or sad face. If you touch a face then it changes that face and all neighbouring faces—those that share an edge—from sad to happy or happy to sad. For example, the right diagram is obtained from the left diagram by touching the top left face.

(i) Beginning with all sad faces, as in the left diagram, is it possible to change this to all happy faces by touching the faces in a particular sequence?

(ii) Beginning with *any* initial setup of happy and sad faces, is it possible to change this to all happy faces by touching the faces in a particular sequence?



7. Six points are drawn in the plane such that no three points lie on a straight line and such that all 15 distances between pairs of points are distinct. The six points form 20 triangles. Prove that among these there is one triangle whose longest side is also the shortest side of another triangle.