The University of Melbourne–Department of Mathematics and Statistics
School Mathematics Competition, 2019
SENIOR DIVISION: SOLUTIONS

1. Let the positive integer N be given by

$$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{N}}}}} = 2019.$$

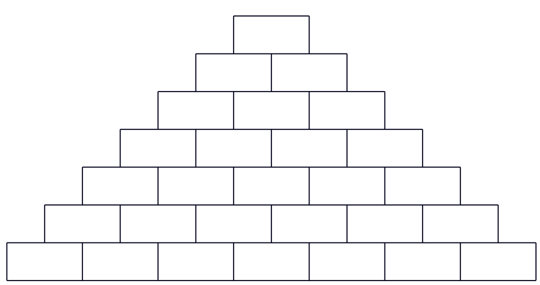
Simplify, as much as possible, the expression

$$\frac{\sqrt{N\sqrt{N\sqrt{N\sqrt{N\sqrt{N}}}}}}{N}.$$

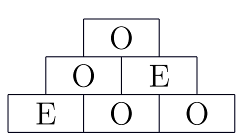
Solution:
$$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{N}}}}} \times \sqrt{N\sqrt{N\sqrt{N\sqrt{N\sqrt{N}}}}} = \sqrt{N\sqrt{N\sqrt{N\sqrt{N\sqrt{N^2}}}}} = \sqrt{N\sqrt{N\sqrt{N\sqrt{N^2}}}} = \sqrt{N\sqrt{N\sqrt{N^2}}} = \ldots = \sqrt{N^2} = N$$

hence $2019 \times \frac{\sqrt{N\sqrt{N\sqrt{N\sqrt{N\sqrt{N}}}}}}{N} = 1$ and $\frac{\sqrt{N\sqrt{N\sqrt{N\sqrt{N\sqrt{N}}}}}}{N} = \frac{1}{2019}$.

2. Maya places integers in each of the boxes below, so that each integer is the sum of the two integers immediately below it. At most how many odd numbers can Maya write?



Solution 1: If two evens appear next to each other, we can replace them with two odds, which does not affect what is above, and does not make worse what is below—one can check this by filling. The strategy is to fill from the top and always place two odds below an even.



Continue in this way.

O
O E
E O O
O O E O
O E O O E
E O O E O O
O O E O O E O

We get 19 odds. One can also fill in this way:

E
O O
O E O
E O O E
O O E O O
O E O O E O
E O O E O O E

which produces only 18 odds. Alternatively, one can show by filling that the 5th, 6th and 7th rows must each have at least two evens, since a single even propagates two many evens (except in the example of 18 odds above). Similarly, the 2nd, 3rd and 4th rows must each have at least one even. Hence there are at most $1+1+2+3+3+4+5=19$ odds, which is realsied above. Hence Maya can write as most 19 odd numbers.

Solution 2. The following solution is due to Grace He. We begin with a few observations. First, consider any three box triangle and observe that it is not possible for all three boxes in such a triangle to contain odd numbers. Thus each three box triangle must contain at least one even number. Now consider any six box triangle. Notice that specifying the parity of the numbers in the bottom three boxes uniquely determines the parity of the numbers in every other box of the triangle, hence there are $2^3=8$ possible ways to fill in a six box triangle:

E
E E
O O O

O
E O
O O E

E
O O
O E O

O
O E
O E E

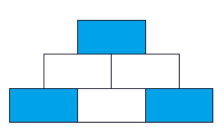
E
E E
E E E

E
O O
E O E

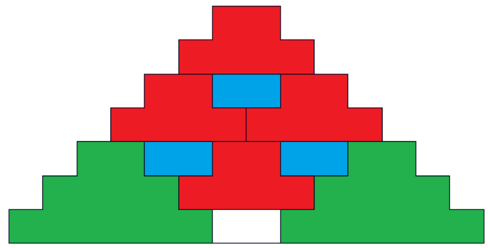
O
O E
E O O

O
E O
E E O

In all cases, at least two of the six boxes contain even numbers. Moreover, we see that at least one of the three corner boxes (coloured in blue below) of any six box triangle must contain an even number.



Now consider splitting the given figure into regions as shown below:



From our prior observations, we know that each of the four red regions must contain at least 1 even number, each of the two green regions must contain at least 2 even numbers, and also that at least one of the three blue boxes must contain an even number. Adding these up, we know that there must be at least $4 \times 1 + 2 \times 2 + 1 = 9$ even numbers, so that there can be at most $28 - 9 = 19$ odd numbers. A configuration which achieves a total of 19 odd numbers is shown in Solution 1.

3. A class of 30 students plays the following game. The numbers 1 to 32 are written on the blackboard. The first student replaces two of the numbers on the blackboard with their sum decreased by 1; the second student replaces two of the numbers on the blackboard with their sum decreased by 2; and so on, so that the nth student replaces two of the numbers on the blackboard with their sum decreased by n. The game continues until each student has played. At the end of the game the final two numbers on the board are both positive. What are all the possible final two numbers on the board?

Solution: The sum of the final two numbers is

$$1 + 2 + ... + 32 - 1 - 2 - ... - 30 = 31 + 32 = 63.$$

If the number 1 was never replaced, then the final two numbers must be 1 and 62. Similarly, the final two numbers can be i and $63 - i$ for $i \in \{1, ..., 32\}$ if i was never replaced. Hence, the final two numbers can be any two positive numbers that add to 63 (regardless of whether a number was left unreplaced throughout).

4. Find all products

$$N \times b_1 b_2 ....b_{2019} \times b_1 b_2 ....b_{2019} = b_1 b_2 ....b_{2019} b_1 b_2 ....b_{2019}$$

where N is an integer and each b_i is a digit, i.e. an integer satisfying $0 \leq b_i < 10$, $b_1 \neq 0$ and we write decimal numbers in terms of digits, so for example $b_1 b_2 = 34$ if $b_1 = 3$ and $b_2 = 4$.

Solution: Put $m = b_1 b_2 ....b_{2019}$. Then we have

$$N \times m^2 = (10^{2019} + 1)m \Rightarrow m = \frac{10^{2019} + 1}{N}$$

In particular , $N \mid 10^{2019} + 1$. If $N > 10$ then $\frac{10^{2019}+1}{N} < 10^{2018}$ so m will have fewer than 2019 digits. Hence N is less than 10, odd and $N \neq 3$, 5 or 9 because $10^n + 1 \equiv 2 \mod 3$ and $10^n + 1 \equiv 1 \mod 5$ for any n. Also, $N \neq 1$ since if $m^2 = (10^{2019} + 1)m$ then $m = 10^{2019} + 1$ which has 2020 digits. Hence $N = 7$.

Now $7 \mid 10^n + 1$ when $n \equiv 3 \mod 6$ hence $7 \mid 10^{2019} + 1$. We divide to get

$$b_1 b_2 ....b_{2019} = 142857142857...142857143$$

where the last digit is 3 since $7 \times 3 \times 3 = 3 \mod 10$, or by rounding up 2.8.

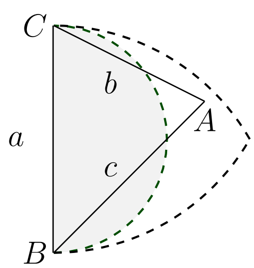
5. Given the longest length of a triangle, if we randomly choose its other two lengths, what is the probability that the triangle is acute? In other words, given a length $a > 0$, randomly choose $0 < b < a$ and $0 < c < a$ such that $b + c > a$, and form a triangle with side lengths a, b and c, then what is the probability that all angles of the triangle are less than $\pi/2$?

This question is ambiguous as stated. The following solutions assume different distributions. The first of these assumes uniform distributions of points, and the second assumes uniform distributions of lengths. (A third solution chooses the lengths consecutively uniformly in one dimension resulting in a non-uniform distribution in the plane.) Suppose the triangle has vertices A, B and C and opposite lengths a, b and c.

Respectively, the solutions assume:

1. A uniform distribution of the point A with fixed B and C in the plane.
2. A uniform distribution of lengths b and c in the square.
3. First choose b uniformly on $[0, a]$ then choose c uniformly on $[a - b, a]$.

Solution 1:



In the picture the interval of length a is one edge of a triangle. We choose the vertex A of the triangle from the closed region given by the intersection of two radius a quarter-circles, so that $0 < b < a$ and $0 < c < a$. The shaded region has boundary a radius $a/2$ circle centred at the midpoint of the vertical edge, and gives those points where the triangle is obtuse. This is because those points on its boundary satisfy $a^2 = b^2 + c^2$.

The area of the closed region is

$$\frac{\pi a^2}{6} + \frac{\pi a^2}{6} - \frac{a^2\sqrt{3}}{4}$$

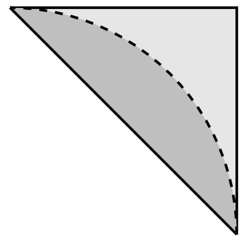
which gives two times the area of $1/6$ of a circle of radius a minus the area of an equilateral triangle of side length a. The shaded region is half of a circle of diameter a and has area

$$\frac{\pi a^2}{8}.$$

Hence the probability of choosing an acute triangle, i.e. the probability of choosing a point in the region bounded by the broken lines, is

$$1 - \frac{\frac{\pi a^2}{8}.}{\frac{\pi a^2}{6} + \frac{\pi a^2}{6} - \frac{a^2\sqrt{3}}{4}} = 1 - \frac{3\pi}{8\pi - 6\sqrt{3}} = \frac{5\pi - 6\sqrt{3}}{8\pi - 6\sqrt{3}}.$$

Solution 2: The following solution is due to Jeff Li. We may assume $a = 1$. Then, b and c are chosen uniformly from the region $0 < b < 1$, $0 < c < 1$ and $b + c > 1$. The acute triangles occur when $b^2 + c^2 > 1$.



So the probability of choosing an acute triangle, i.e. the probability of choosing a point in the lightly shaded region, is given by the area of the lightly shaded region divided by the total area of both regions.

$$P = \frac{1-\pi/4}{1/2} = 2-\pi/2.$$

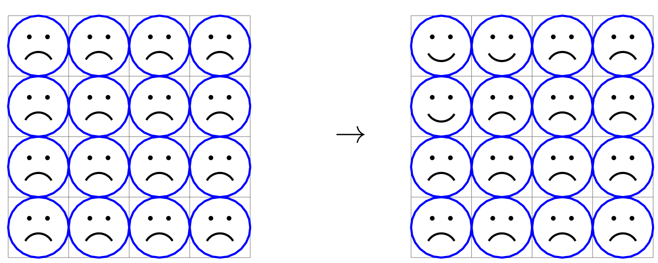
Solution 3: The following solution is due to Jeff Li. Again assume $a=1$. Choose b uniformly from $(0,1)$. For a given b, choose c uniformly from $(1-b,1)$, then the triangle is acute only if c is in $(\sqrt{1-b^2},1)$. So the probability that the triangle is acute is $P(b)=(1-\sqrt{1-b^2})/(1-(1-b)$. The probability of choosing an acute triangle probability integrates $P(b)$:

$$P = \int_0^1 P(b)db = \int_0^1 \frac{1-\sqrt{1-b^2}}{b}db = 1-\ln(2).$$

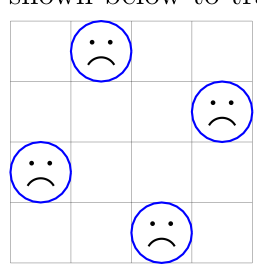
6. Consider a 4×4 grid with each position containing a happy or sad face. If you touch a face then it changes that face and all neighbouring faces—those that share an edge—from sad to happy or happy to sad. For example, the right diagram is obtained from the left diagram by touching the top left face.

(i) Beginning with all sad faces, as in the left diagram, is it possible to change this to all happy faces by touching the faces in a particular sequence?

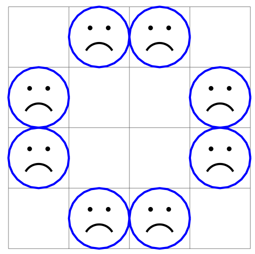
(ii) Beginning with *any* initial setup of happy and sad faces, is it possible to change this to all happy faces by touching the faces in a particular sequence?



Solution: Simply touch the four faces shown below to transform from all sad to all happy.



No. There are initial setups of happy and sad faces, from which it is impossible to change to all happy faces. There are 2^{16} possible configurations of happy and sad faces. There are also 2^{16} possible ways to touch faces, since the order of touching does not matter, and we need to touch a face at most once because touching twice is the same as not touching. If we touch the faces in the diagram below, then we will return to the original position, so it is the same as touching no faces. But this means that from all happy faces, we can only reach fewer than 2^{16} configurations, and conversely there are initial setups of happy and sad faces, from which it is impossible to change to all happy faces.



7. Six points are drawn in the plane such that no three points lie on a straight line and such that all 15 distances between pairs of points are distinct. The six points form 20 triangles. Prove that among these there is one triangle whose longest side is also the shortest side of another triangle.

Solution 1: Colour each longest side of a triangle red. We will prove that there is a triangle with all sides red, hence its shortest side is also the longest side of another triangle.

Label the closest two points P_1 and P_2. The 4 triangles containing $\overline{P_1P_2}$ produce 4 red edges incident to either P_1 or P_2. Either

(i) there are at least 3 red edges incident to one of P_1 and P_2, or

(ii) P_1 and P_2 are each incident to 2 red edges.

In case (i) the other ends of the three red edges incident to P_1, say, form a triangle, one of whose edges is red, and this together with two of the red edges incident to P_1 forms a triangle with all sides red. In case (ii), label the other ends of the three red edges incident to P_2 by P_3 and P_4, so $\overline{P_2P_3}$ and $\overline{P_2P_4}$ are red. Then $\triangle P_1P_3P_4$ forces either $\overline{P_3P_4}$ to be red, hence $\triangle P_2P_3P_4$ has all sides red as desired, or one of $\overline{P_1P_3}$ or $\overline{P_1P_4}$ to be red, in which case P_1 is incident to 3 red edges which reduces to case (i).

Solution 2: The following solution is due to Jeff Li. For every edge joining a pair of points, colour it red if it is the longest side of some triangle, otherwise colour it blue. We will first show that there is a monochromatic triangle, i.e. a triangle with all sides the same colour.

Indeed, pick any point, call it P_1, and consider the five edges incident to this point. By the Pigeonhole Principle, three of these five edges must be the same colour. Label the points at the other ends of these three edges P_2, P_3 and P_4. If the edges P_2P_3, P_2P_4 and P_3P_4 were all the same colour, then $\triangle P_2P_3P_4$ would be a monochromatic triangle. On the other hand, if these three edges were not all the same colour, then one of them would be the same colour as P_1P_2, P_1P_3 and P_1P_4 so would form a monochromatic triangle with two of these edges. In both cases, there is a triangle with all sides the same colour.

Now take this monochromatic triangle and notice that it must have at least one red side since its longest side is coloured red. Then, since the triangle is monochromatic, the shortest side of this triangle must also be coloured red, hence the shortest side of this triangle is also the longest side of some other triangle.